Quantitative Finance Qualifying Exam

May 2022

Instructions: (1) You have 4 hours to do this exam. (2) This exam is closed notes and closed books. No electronic devices are permitted. (3) Phones must be turned completely off during the exam. (4) All problems are weighted equally.

Part 1: Do 2 out of problems 1, 2, 3. (AMS511)
Part 2: Do 2 out of problems 4, 5, 6. (AMS512)
Part 3: Do 2 out of problems 7, 8, 9. (AMS513)
Part 4: Do 2 out of problems 10, 11, 12. (AMS517)

Problems to be graded: Please write down which eight problems you want graded here.

Academic integrity is expected of all students at all times, whether in the presence or absence of members of the faculty. Understanding this, I declare that I shall not give, use, or receive unauthorized aid in this examination.

Name (Print clearly):

Student ID:

Signature:

Stony Brook University
Applied Mathematics and Statistics

1. An asset's price $S(t)$ follows the dynamics of the constant coefficient geometric process described by the stochastic differential equation (SDE):

$$dS(t) = \mu\, S(t)\, dt + \sigma\, S(t)\, dW(t).$$

Let $Z(t) = \log[S(t)]$. Apply It's lemma to calculte the dynamics of $Z(t)$.

2. An investment universe of equities has returns which follow a multivariate Gaussian distribution with mean vector $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma}$. Let r_f denote the risk free rate of return and $\mathbf{x}$ the allocation vector. Assume that the notional total capital is 1 and that short positions are permitted. Derive a closed form expression which computes the mean-variance (Markowitz) portfolio representing the tangent portfolio. Show all work supporting your answer.

3. John loans Mary \$10,000 at an annually compounded interest rate of 6%. Mary desires to pay John the principal and accrued interest at the end of 3 years

   - What is the amount due John from Mary at the end of 3 years.
   - John instead wants Mary to make 3 equal annual payments at the same interest rate. What is the amount of that payment.

4. A distribution is said to have a power law tail if its survival function has the form:

$$Prob[R > r] = 1 - F(r) = L(r)r^{-\alpha}, \quad \alpha > 0$$

where $F(r)$ is the cumulative distribution of the return R and $L(r)$ is a slowly varying function such that

$$\lim_{r \to \infty} [L(\lambda r) \,/\, L(r)] = 1, \quad \forall \lambda > 0$$

- For a return distribution with a power law tail, demonstrate mathematically which moments of R ($E\left[R^i\right], i = \{1, 2, 3, \ldots\}$) exist depending upon the value of the tail exponent.
- Given a sample of data, describe how an appropriately constructed plot of the survival function can be used to identify if a power law tail appears to exist and, if so, how that plot can be used to estimate the tail exponent α.

5. You are given the returns for $N = 10$ assets over $T = 60$ time periods. You wish to examine the sample correlaton matrix.

   - Compute the parameter q for the Marchenko-Pastur distribution of eigenvalues for a correlation matrix of uncorrelated assets for a problem of this dimension and sample size.
   - Compute the lower and upper bound for the associated Marchenko-Pastur distribution.
   - You are given the following partial list of sorted eigenvalues of the sample correlation matrix: $\{10.1, 7.4, 6.2, 5.5, 3.1, 1.8, 0.9, 0.8, 0.4, 0.2\}$. Based solely on the distribution and without any adjustment for sample size, which eigenvalues appear to be statistically meaningful.
   - Using this information, denoise the eigenvalues. What are the values of the eigenvalues not found to be statistically meaningful.

6. You are given three stocks, $i = \{1, 2, 3\}$. Each stock's return r_i can be modeled by the Captial Asset Pricing Model (CAPM):

$$r_i - r_f = \beta_i(m - r_f) + \epsilon_i.$$

Given $r_f = 0.03$, $\beta = \{0.8, 1.0, 1.2\}$, $\mu_m = 0.085$, $\sigma_m^2 = 0.01$, and $\sigma_e^2 = \{0.022, 0.0011, 0.009\}$

- What is the mean vector and covariance matrix for the three stocks?
- Assuming short positions are permitted and no constraints, derive to proportiona lity a compute solution for the mean-variance portfolio optimization problem.

7. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $\{W_t : t \geq 0\}$ be the $\mathbb{P}$-standard Brownian motion with respect to the filtration $\mathcal{F}_t$, $t \geq 0$.

   Define $X_t = x_0 + W_t$ where $x_0 \in \mathbb{R}$, and the following three stopping times:

   - $T_a = \inf\{t \geq 0 : X_t = a\}$;
   - $T_b = \inf\{t \geq 0 : X_t = b\}$, where $a < x_0 < b$; and
   - $T = \inf\{t \geq 0 : X_t \notin (a, b)\}$, where $a < x_0 < b$.

   (i) Show that $\{X_t : t \geq 0\}$ is a continuous martingale
   (Show the three properties $\mathbb{E}[X_t \mid \mathcal{F}_s] = X_s$, $\mathbb{E}[|X_t|] < \infty$, and $X_t \in \mathcal{F}_t$.)

   (ii) Show, using the optional stopping theorem, that

   $$\mathbb{P}(T_a < T_b) = \frac{b - x_0}{b - a}$$

   (iii) Show that $\left\{Y_t = (X_t - x_0)^2 - t : t \geq 0\right\}$ is a continuous martingale.
   (Show the three properties $\mathbb{E}[Y_t \mid \mathcal{F}_s] = Y_s$, $\mathbb{E}[|Y_t|] < \infty$, and $Y_t \in \mathcal{F}_t$.)

   (iv) Assuming that $T < \infty$ almost surely show, using the optional stopping theorem, that

   $$\mathbb{P}(X_T = a \mid X_0 = x_0) = \frac{b - x_0}{b - a} \quad \text{and} \quad \mathbb{P}(X_T = b \mid X_0 = x_0) = \frac{x_0 - a}{b - a}$$

   with

   $$\mathbb{E}(T) = (b - x_0)(x_0 - a).$$

8. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $\{W_t : t \geq 0\}$ be the $\mathbb{P}$-standard Brownian motion with respect to the filtration $\mathcal{F}_t$, $t \geq 0$. Suppose $S_t > 0$ follows a constant elasticity of variance (CEV) model of the form

$$dS_t = rS_t dt + \sigma(S_t, t)S_t dW_t$$

where r is a constant and $\sigma(S_t, t)$ is a local volatility function.

By setting $\sigma(S_t, t) = \alpha S_t^{\beta-1}$ with $\alpha > 0$ and $0 < \beta < 1$, show using Itô's formula that $\sigma(S_t, t)$ satisfies

$$d\sigma(S_t, t) = \sigma(S_t, t)(\beta - 1)\left\{\left(r + \frac{1}{2}(\beta - 2)\sigma(S_t, t)^2\right) dt + \sigma(S_t, t)dW_t\right\}.$$

Finally, conditional on S_t show that for $t < T$,

$$S_T = e^{rT}\left[e^{-r(1-\beta)t}S_t^{(1-\beta)} + \alpha\int_t^T e^{-r(1-\beta)u}dW_u\right]^{\frac{1}{1-\beta}}$$

(Hint: Apply Itô lemma to $X_t = e^{-rt}S_t$.)

Prove that $\widetilde{P}$ is a probability measure.

9. Let $\{W_t : t \geq 0\}$ be the $\mathbb{P}$-standard Brownian motion on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Consider a market that consists of a risk-free asset and a stock, whose values at time t are B_t and S_t, respectively. Assume that these values evolve according to the following processes

$$dB_t = rB_t dt \quad \text{and} \quad dS_t = (\mu - D)S_t dt + \sigma S_t dW_t$$

such that D is the continuous dividend yield, r is the risk-free rate, μ is the stock price growth rate and σ is the stock price volatility.

Show that under the no arbitrage assumption, the price of the American option $V(S_t, t)$ has to satisfy the following inequality

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V}{\partial S_t^2} + (r - D)S_t \frac{\partial V}{\partial S_t} - rV(S_t, t) \leq 0$$

with constraint

$$V(S_t, t) \geq \psi(S_t)$$

where $\psi(S_t)$ is the intrinsic value of the American option at time t.
Hints:

- Consider a Δ-hedged portfolio: $\Pi_t = V(S_t, t) - \Delta S_t$, then (because of the continuous dividend) $d\Pi_t = dV(S_t, t) - \Delta(dS_t + DS_t dt)$;
- using Taylor's expansion and subsequently using Itô lemma compute $dV(S_t, t)$, and eliminate the random component in $d\Pi_t$ by choosing an appropriate Δ;
- use the remaining terms in $d\Pi_t$ and the no arbitrage argument to justify the inequality.

10. Consider the Franck copula

$$C(u_1, u_2) = \frac{1}{\theta} \log \left(1 + \frac{(e^{\theta u_1} - 1)(e^{\theta u_2} - 1)}{e^{\theta} - 1} \right)$$

where θ may be any real number. (a) Check that the copula conditions are satisfied. (b) Show that the Franck copula converges to the independent copula as $\theta \to 0$.

11. Consider the GARCH(1,1) model $(\alpha, \beta, \alpha + \beta \in (0, 1))$

$$z_t = \sigma_t \epsilon_t, \quad \sigma_t^2 = \omega + \alpha z_{t-1}^2 + \beta \sigma_{t-1}^2$$

Show that the kurtosis of the observations z_t are larger than 3.

12. Suppose returns y_t of an asset follow the ARMA(1,1)-GARCH(1,1) model $(a, b \in (-1,1), a \neq 0, \alpha \in (0,1))$

$$y_t = a y_{t-1} + z_t + b z_{t-1}, \qquad z_t = \sigma_t \epsilon_t, \qquad \sigma_t^2 = \omega + \alpha z_{t-1}^2,$$

where ϵ_t are independent and identically distributed standard normal random variables with mean 0 and variance 1. Compute 95% 1-day and 5-day VaR and ES for the long position of the asset?