Loading particles into the wakefield absorbs energy from the wake. The absorbed energy is manifest in the reduction of the electric field available for accelerating the electrons. This effect is called beam loading. We have two objectives in beam loading:

1. Load as much charge as possible to increase the accelerated beam brightness
2. Minimize the energy spread of the accelerated beam. Since we are intending to inject substantial amount of charge into the wakefield, these charges are going to have a finite size both longitudinally and transversely. Therefore we want these electrons to be loaded in a way that the accelerating field felt by the entire beam is the same. Doing so will ensure a small final energy spread for the beam.

We will treat the two cases of linear and nonlinear beam loading separately.

We start from the discussion of the linear theory. We will the provide a physical picture of beam loading in the nonlinear regime and discuss the advantages of this regime. The analytical calculations are presented and the efficiency and beam quality are addressed.

Linear Theory

The impact of loading a trailing particle beam in the plasma wakefield in linear theory can be calculated using the superposition principle; i.e. the wakefield due to the driver and trailing beam is separately calculated and the total wakefield is the sum of the fields of the individual beams.

We know that the wake function behind the driver in the 1D linear regime has a sinusoidal form:

$$\psi = \psi_0 \sin(\xi) \quad \ldots (1)$$

$$E = \frac{\partial \psi}{\partial \xi} = \psi_0 \cos(\xi) \quad \ldots (2)$$

The equations above are in normalized units. From the normalized relationship between the wake potential, ψ & the density perturbation $\frac{n_1}{n_0}$ for a wide driver, we can get the amplitude in terms of density perturbation:

$$(\nabla_\perp^2 - 1)\,\psi = \chi \quad \ldots (3)$$ ← eqn 42 from the laser & beam coupling to plasma lecture

$$\chi = \frac{n_1}{n_0} - \phi_p$$

In 1D analysis, $\nabla_\perp \to 0$

Since we are interested in beam loading, we are looking at a region behind the driver, & $\phi_p = 0$

$$\Rightarrow (3) \Rightarrow \psi_0 = \frac{n_1}{n_0}$$ ← amplitude of density perturbation

$$\Rightarrow (2) \Rightarrow E = \frac{n_1}{n_0}\cos(\xi) \quad \ldots (4)$$

$$(4) \text{ unnormalized} \Rightarrow \frac{eE}{mc\omega_p} = \frac{n_1}{n_0}\cos(k_p\xi) \quad \ldots (5)$$

We can find the maximum number of charge we can load in the field if the wakefield generated by our loaded particles cancel the wake from the driver. Consider an ultrashort beam of N_0 electrons loaded at the minimum of the accelerating wave, which we will require to create a wave equal in amplitude to that of the initial wakefield, so they cancel, i.e. all the energy in the wake behind the drive beam is absorbed by the trailing beam.

The wake of a thin sheet of charge with surface charge density $\sigma = q/A$ was derived at the end of the "laser and beam coupling to plasma" lecture and is given by

$$E_z = -\underbrace{\eta(\xi-\xi_0)}_{\text{Heaviside step fcn}}\cos(\xi-\xi_0)\,\sigma \quad \ldots (6)$$

$$(6) \Rightarrow \text{unnormalized: } \frac{eE}{mc\omega_p} = -\eta\left(k_p(\xi-\xi_0)\right)\cos\left(k_p(\xi-\xi_0)\right)\frac{q}{A}\cdot\frac{\overset{\omega_p/c}{\cancel{k_p}}}{en_0}$$

$$\text{unnormalizing } q\text{: } \quad q \sim \rho V = \frac{\rho}{en_0}\cdot \bar{V} k_p^3$$

$$\Rightarrow E = -\eta(\xi-\xi_0)\cos\left(k_p(\xi-\xi_0)\right)\frac{q}{A}\;\overset{''\left(\frac{e^2 n_0}{m\epsilon_0}\right)}{\omega_p^2}\cdot\frac{m}{e^2 n_0}$$

$$\Rightarrow E = -\frac{q}{\epsilon_0 A}\,\eta(\xi-\xi_0)\cos\left(k_p(\xi-\xi_0)\right) \quad \ldots (7)$$

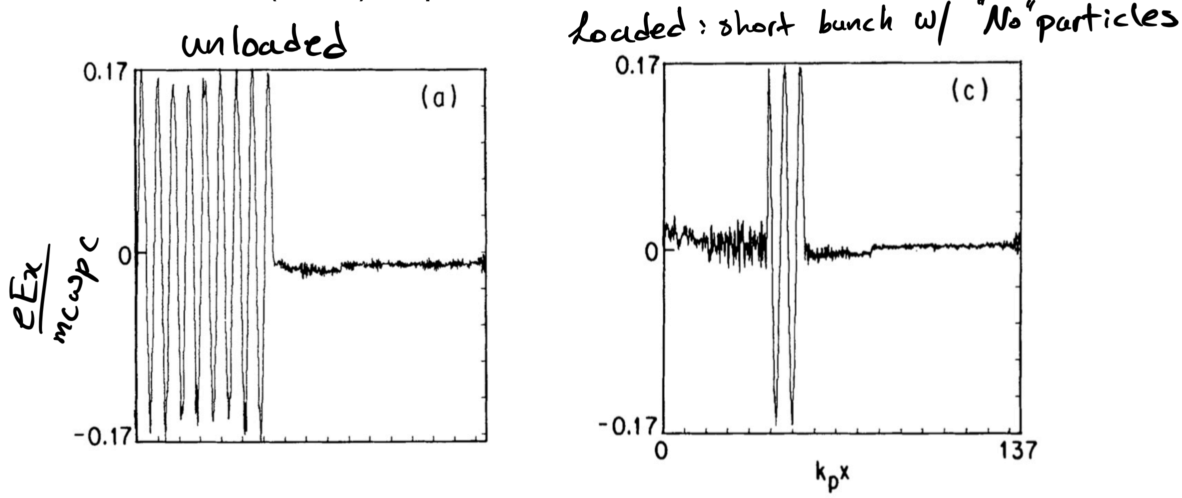
Comparing ⑦ & ⑤, for E field to cancel, the amplitudes have to match, meaning that

$$\frac{q}{A\epsilon_0} = \frac{n_1}{n_0}\, mc\omega_p/e$$

$$\Rightarrow q = eN_0 \Rightarrow N_0 = \frac{n_1}{n_0} A \overbrace{\frac{m\epsilon_0}{n_0 e^2}}^{1/\omega_p^2} c\omega_p\, n_0$$

$$\Rightarrow N_0 = \frac{n_1}{n_0} A \frac{n_0}{k_p} \approx 5\times10^5 \left(\frac{n_1}{n_0}\right)\sqrt{n_0[cm^{-3}]}\; A[cm^2] \quad \ldots (8)$$

Recall that for a 1D wake, ‘A’ represent the cross sectional area of the wake as well as the driver.
The result of 1D simulation showing the cancellation of the initial wakefield by the wake due to the trailing beam is shown as Figure 1 in Katsouleas, particle accelerators, **22**, 81 (1987), reproduced below



Unfortunately in this case, ideal beam loading is achieved at the cost of 100% energy spread. This is because the front of the beam experiences the full field of the wake, while the back of the beam experiences zero field!

A ramped charge density can be used to flatten the wake, i.e. create a constant electric field within the bunch. Particles within such a particle bunch get accelerated uniformly, creating a very small energy spread. The appropriate shape of the density profile, which happens to be a ramped beam can be obtained from the superposition principle and the requirement that the wake fields be constant inside the beam:

$$E_t = \underbrace{E_0 \cos\xi}_{\text{intial wake (eqn 4)}} - \underbrace{\int_{\xi_0}^{\xi} d\xi' \, \rho(\xi') \cos(\xi - \xi')}_{\text{wake generated by beam}} \quad \cdots (9)$$

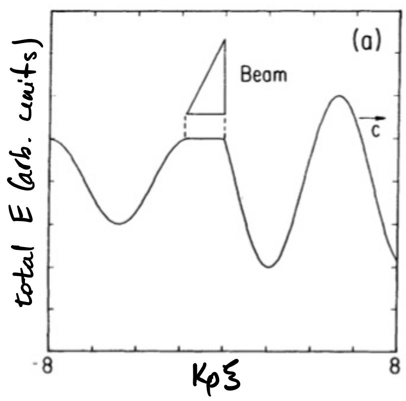
where ξ_0 is the front of the beam & E_t is the total field of the wake & the beam & is constant.

Assuming that the bunch density $\rho_b(\xi)$ has the form $a\xi + b$, begining from $\xi = \xi_0$ at the wave's minimum, and solving for a & b, we get

$$\rho_b(\xi) = - E_0 \left[(\cos\xi_0)\, \xi + (\sin\xi_0 - \xi_0 \cos\xi_0) \right] \quad \cdots (10)$$

Recall that in normalized units, $E_0 = n_1$, the density perturbation.

The results were shown in Fig 4(a) of the 1987 Katsoleaus paper reproduced here.



The maximum charge is obtained by letting ρ_b go to zero (we restrict ρ_b to have only one sign)

The beam's properties in normalized units are:

peak beam density: $\rho_b(\xi_0) = \rho_b^{max} = -n_1 \sin\xi_0$

Max bunch length: $l_{max} = \tan\xi_0$

Accelerating field: $E_t = E_0 \cos(\xi_0)$

# of particles: $N = N_0 \dfrac{\sin^2\xi_0}{2\cos\xi_0}$

where N_0 is given by Eqn 8.

while this solutions, there is compromise between maximum charge, accelerating field, and efficiency

Energy absorbed per unit length:

$$\left.\begin{aligned} &E_t Q_b = E_0 \cos(\xi_0)\, N_0 \frac{\sin^2\xi_0}{2\cos\xi_0} \\ &N_0 = n_1 A = E_0 A \\ &\sin^2\xi_0 + \cos^2\xi_0 = 1 \end{aligned}\right\} \Rightarrow E_t Q_b = \underbrace{\frac{E_0^2}{2}}_{\text{Def'n } \mathcal{E}_0 \text{ (energy per unit length in front)}} A - \underbrace{\frac{E_t^2}{2}}_{\text{Def'n } \mathcal{E}_t \text{ (energy per unit length behind the bunch)}} A$$

Note that the amplitude of the wakefield behind the bunch is reduced to E_t
The beam loading efficiency is then given by how much of the energy of the wake is absorbed by the trailing beam:

$$\eta_L = 1 - \frac{\mathcal{E}_t}{\mathcal{E}_0} = \begin{cases} 100\% & \text{for } \mathcal{E}_t = 0 \text{ (zero accelerating field)} \\ 0\% & \text{for } \mathcal{E}_t = \mathcal{E}_0 \text{ (max field)} \end{cases}$$

It is clear that in the linear regime, you either have to choose to accelerate your beam at high efficiency or at a large gradient.

Constraints on width: in finite width linear wave, acceleration field changes with width. The focusing and defocusing forces are also nonlinear for the small amplitude waves. This result in a non uniform beam quality for a trailing beam with width>0. One solution is to use a matched beam, where the focusing force is matched by the beam emittance

for $\epsilon_N \sim \mu m$ (collider quality), we need

$k_p \sigma_r \ll 1$ to match, $F_\perp \sim r$

Matching and the effect of linear focusing will be discussed in a future lecture. Here, the important point is that we desire the beam to be narrow, and from the previous lecture we know that the narrow beam interacts with and absorbs the wake out to a skin depth, I.e. $A_{eff} \sim c^2/\omega_p^2$

This value for the area allows us to estimate the number of particles in the beam:

$$\frac{1}{k_p} \equiv c/\omega_p \simeq 5.3\times10^5 / \sqrt{n_0[\text{cm}^{-3}]}$$

Max particles: $N_0 \simeq \left(\frac{n_1}{n_0}\right) n_0 k_p^{-3} \simeq \frac{1.5\times10^8}{\sqrt{n_0\,[10^{18}\,cm^{-3}]}} \left(\frac{n_1}{n_0}\right)$ fractional density perturbation

$\frac{n_1}{n_0} \ll 1$ – Linear theory assumption

Even setting $\frac{n_1}{n_0} = \frac{\delta n}{n} \sim 1$ to get an upper limit for N_0, we get

$$N_0 \sim 80\ pC \quad \text{for } n_0 = 10^{17}\,cm^{-3}$$

$$N_0 \sim 240\,pC \quad \text{for } n_0 = 10^{16}\,cm^{-3}$$

This is still clearly an overestimate, because we are assuming all the energy is absorbed. Also, this amount of charge is still relatively small. For HEP applications, we need nC of charge.

In order to increase the loaded charge, we need to increase $\delta n/n_0$ & A_{eff} To non-linear territory.
We will show next that 3D nonlinear wakes have ideal properties for loading and accelerating electrons.

<u>Physical Picture, Tzoufras, Physics of Plasmas 16, 056705 (2009):</u>
The ion channel of the blowout regime is described by the trajectory of the innermost electron of the sheath, i.e. by $r_b(\xi)$, $\xi = ct - z$, where the driver is moving towards positive "z". This description works well except for the very front and very back of the bubble, where electron trajectories cross.

designate $R_b = \max(r_b(\xi))$

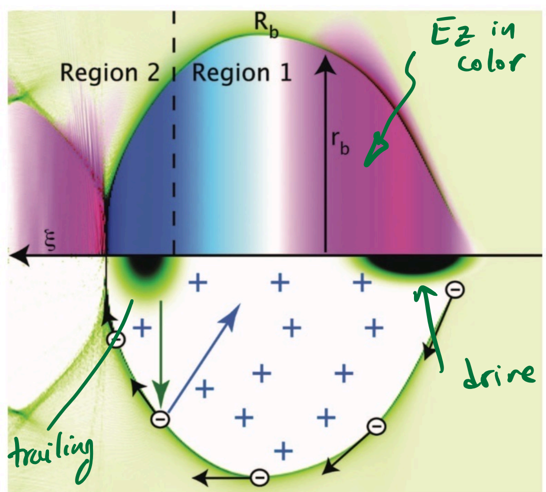
We use the usual normalized units

Length: $\frac{\omega_p}{c} d \rightarrow d$

density: $\frac{n}{n_0} \rightarrow n$

charge: $\frac{q}{e} \rightarrow q$

fields: $\frac{eE}{mc\omega_p} \rightarrow E$



E_z normalized is described by r_b & its derivative:

$$E_z(\xi, r) = E_z[r_b(\xi)] = \frac{d}{d\xi}\left[\frac{1}{4}[1+\beta(r_b)]\, r_b^2\right]$$

weak function of r_b

depends on width of plasma sheath

In the ultrarelativistic limit, where the maximum blowout radius greatly exceeds the skin depth, the trajectory of the inner most electron is described by

$$r_b \frac{d^2 r_b}{d\xi^2} + 2\left(\frac{dr_b}{d\xi}\right)^2 + 1 = \frac{4\lambda(\xi)}{r_b^2} \quad \ldots (11)$$

$$E_z(\xi, r) = \frac{1}{2} r_b \frac{dr_b}{d\xi}$$

note: this is independent of r.

$$\lambda(\xi) = \int_0^\infty q\, r\, n_b(r)\, dr$$

(normalized) charge per unit length, here interested in trailing beam

for Laser, you would add a term dependent on ponderomotive force.

Unlike linear theory, an ultrashort bunch cannot absorb all the energy in the wake. This follows the relation between the sheath and the field. If an ultrashort bunch is loaded at the position of the dashed line to absorb all the energy of the wake, the field would have to become instantaneously zero. This means that the sheath would almost have to follow the dashed line itself. But such a large negative charge would actually repel sheath electrons and delay their crossing of the field. The presence of negative charge will cause the sheath to bend away from the axis, not towards it. Therefore such an event (absorption of all wake energy by an ultrashort bunch) is not possible.

On the other hand, a bunch with finite length and charge per unit length allows the electrons in the sheath to reach the axis while slowly decreasing their transverse momentum. Ideally the electrons should arrive on axis with no transverse momentum, which implies zero longitudinal momentum as well (why? HW). This configuration leads to nearly 100% absorption of the energy available in the bubble.

Energy Considerations in the blowout regime

Consider eqn (11) in a region w/o beam:

$$r_b r_b'' + 2r_b'^2 + 1 = 0 \quad \ldots (12)$$

Using the insight that $r_b'(\xi)\frac{dr_b'}{dr_b} = \cancel{r_b'}\frac{dr_b'}{d\xi}\cdot\frac{\cancel{d\xi}}{\cancel{dr_b}} = \frac{dr_b'}{d\xi} = r_b''$,

eqn 12 can be rearranged & integrated to give

$$r_b^4(2r_b'+1) = \text{Constant}.$$

$\therefore$ we define a constant in the beam-free region given by

$$I_0 = \frac{\pi}{16} r_b^4\left[1 + 2\left(\frac{dr_b}{d\xi}\right)^2\right] \quad \ldots (13)$$

Since we know at the location of max bubble radius, $r_b = R_b$ & $\frac{dr_b}{d\xi} = 0$, we can associate I_0 with a max bubble radius given by

$$\Rightarrow \boxed{I_0 = \frac{\pi}{16} R_b^4} \quad \ldots (14)$$

Also note,

$$\frac{dI_0}{d\xi} = \frac{\pi}{4} r_b^3 \frac{dr_b}{d\xi}\left[r_b\frac{d^2r_b}{d\xi^2} + 2\left(\frac{dr_b}{d\xi}\right)^2 + 1\right] \quad \ldots (15)$$

So when $\lambda \neq 0$,

$$\frac{dI_0}{d\xi} = -\frac{\pi}{2} r_b^3 \frac{dr_b}{d\xi}\left[\frac{2\lambda(\xi)}{r_b^2}\right] \quad \ldots (16)$$

If we integrate this expression between ξ_0 & ξ, and using the Expression for E_z from previous page, for a region of beam-loaded electrons, we get

$$I_0(\xi) - I_0(\xi_0) = -\underbrace{2\pi}_{=\int_0^{2\pi} d\phi} \int_{\xi_0}^{\xi}\int_0^{r_b} E_z(\xi')\,\rho_B(\xi', r')\,r'\,dr'\,d\xi' \quad \ldots (17)$$

This expression allows us to understand the physical interpretation of I_0.
The integral on the right hand side is essentially $q\int \vec{J}_B \cdot \vec{E}\, dV$, which is the power of the energy exchange between the field of the bubble & the electron bunch in the volume bounded by ξ_0 & ξ.

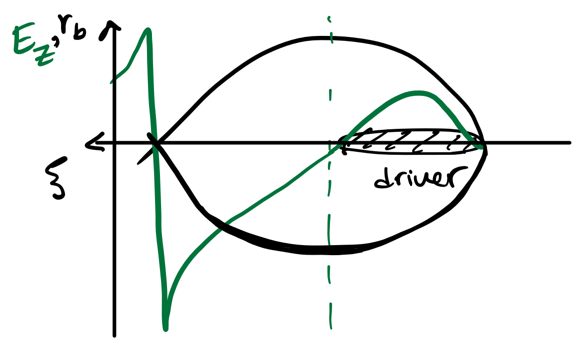
Recall that the relativistic energy conservation is given by

$$\frac{d}{dt}(\gamma mc^2) = q(\vec{E}\cdot\vec{v}) \quad \text{for a charge } q$$

$$dq = \rho\, dV \implies \int \vec{E}\cdot\vec{v}\,\rho\, dV$$

$$= \int \vec{E}\cdot\vec{J}\, dV \; : \text{ rate of energy exchange for the bunch}$$

Therefore, I_0 serves as a measure of energy density in the bubble.



As the drive bunch generates the wake, I_0 increases since in that region, $E_z > 0$. This means that the beam is pumping energy into the wake, resulting in correspondingly larger & larger max bubble radius, R_b. On the other hand, when the trailing beam is accelerated by the wake, $E < 0$ & I_0 decreases. Therefore after the trailing beam, the trajectory of r_b would follow that corresponding to a bubble with $\tilde{R}_b < R_b$ since

$$I_0 = \frac{\pi}{16} R_b^4$$

$$\tilde{I}_0 = I_0 \text{ after the beam} < I_0$$

$$\boxed{\tilde{R}_b < R_b} \quad \cdots (18)$$

This formalism was worked out by A. Golovanov, PPCF, 63. (2021).

Reminder on the properties of blowout:

- Linear focusing force and constant accelerating field
- Beam's self fields cancel to the order of $1/\gamma^2$
- Forces for a cylindrically symmetric wake are

$$\left.\begin{aligned} F_\perp &= \nabla_\perp \psi = -r/2 \\ F_z &= -\frac{\partial \psi}{\partial \xi} \end{aligned}\right\} \Rightarrow \frac{\partial F_\perp}{\partial \xi} = \frac{\partial F_z}{\partial r} = 0 \quad \text{(Panofsky–Wenzel)}$$

recall, these are normalized!

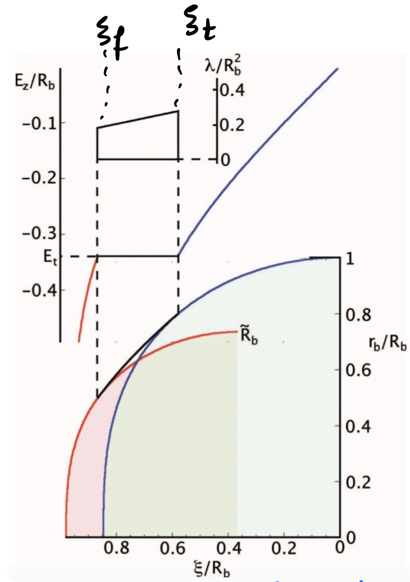
wake potential $\psi = \phi - A_z = [(1+\beta)\, r_b^2(\xi) - r^2]/4$

Variation in accelerating force can cause significant energy spread for a beam whose profile is not properly tailored. First we discuss the physical picture of wake flattening this is accomplished.

Consider a trailing beam placed inside a wakefield with initial maximum bubble radius of R_b. To avoid nonlinear focusing fields, we require that this beam fit inside the bubble, i.e.

Def'n: $r_b(\xi_f) \equiv r_f$

Spot size: $r_w < r_f$



blue: absence of e^- bunch

In presence of the trailing bunch, the trajectory is modified to correspond to the black curve in the region of the beam.

The sheath bends slightly upwards.

The sheath after trailing beam feel only the ion column, but because energy was absorbed, the trajectory in that segment corresponds to a bubble with smaller maximum radius: $\tilde{R}_b < R_b$

<u>Analytical solutions:</u>

Start from the equation for the blowout radius:

$$r_b \frac{d^2 r_b}{d\xi^2} + 2\left(\frac{dr_b}{d\xi}\right)^2 + 1 = \frac{4\lambda(\xi)}{r_b^2} \quad \ldots (11)$$

This equation is valid for $R_b \gg 1$

The idea is to find an expression for Ez by integrating this equation for a bunch with arbitrary current profile. We then study a flat top (for which we know λ) and then find the λ For which the field is constant.

define $\xi = 0$ at $r_b = R_b$, i.e. $\frac{dr_b}{d\xi} = 0$. These will be the initial conditions. We also assume that $r_b(\xi)$ decreases monotonically for $\xi > 0$.

We can express λ as a function of r_b instead of ξ

$$\Lambda(r_b) = \lambda[\xi(r_b)] \quad \dots \text{(19)}$$

Rewrite (11) as

$$r_b'' = \frac{4\Lambda(r_b) - r_b^2[2(r_b')^2 + 1]}{r_b^3} \quad \dots \text{(20)}$$

substituting $r'' = r_b'\left(\frac{dr_b'}{dr_b}\right)$ & integrating (see Appendix of Tzoufras 2009), we obtain

$$E_z = \frac{1}{2} r_b r_b' = -\frac{r_b}{2\sqrt{2}}\sqrt{\frac{16\int^{r_b}\Lambda(\tau)\tau\,d\tau + C}{r_b^4} - 1} \quad \dots \text{(21)}$$

integration constant C is determined by requiring E_z to be continuous

Flat Top Profile

$$\lambda(\xi) = \Lambda(r_b) = \begin{cases} 0 & 0 \leq \xi \leq \xi_t \\ \Lambda_0 & \xi_t \leq \xi \leq \xi_F \end{cases}$$

Boundary condition: $\xi = 0 \Rightarrow r_b(\xi = 0) = R_b, \quad r_b'(\xi = 0) = 0$

$$E_z(\xi) \approx \begin{cases} -\dfrac{r_b}{2\sqrt{2}}\sqrt{\dfrac{R_b^4}{r_b^4}-1} & 0 \le \xi \le \xi_t \quad \cdots (22a) \\ -\dfrac{r_b}{2\sqrt{2}}\sqrt{\dfrac{8\Lambda_0 r_b^2 + C}{r_b^2}-1} & \xi_t \le \xi \le \xi_F \quad \cdots (22b) \end{cases}$$

$$C = R_b^4 - 8\Lambda_0 r_t^2 \quad \cdots (23)$$

$r_t = r(\xi_t)$

To find the trajectory of $r_b(\xi)$, the differential equation

$$E_z = \frac{1}{2} r_b r_b'$$

needs to be solved using the expression derived above.

Before the trailing beam ($0 \le \xi \le \xi_t$), the expression for r_b can be found using incomplete elliptic integrals.

At the location of the beam ($\xi_t \le \xi \le \xi_F$), the sign of C needs to be known to determine behavior of r_b

If $C < 0$, $\exists\, r_b$ s.t. $\dfrac{8\Lambda_0 r_b^2 + C}{r_b^4} - 1 = 0$

$\Rightarrow E_z[r_b(\xi_m)] = 0$

Transverse momentum of innermost e^- changes sign $\rightarrow$ beam loading must stop before ξ_m otherwise r_b is not monotonic

If $C > 0$, $r_b(\xi)$ remains monotonic until $r_b \rightarrow 0$

The larger the C, the more quickly particle reaches ξ axis.

For a fixed r_t, Larger $C \rightarrow$ Lower Λ_0,
Less opposition to sheath e^- going to axis

$\therefore$ Maximum bunch charge that can be loaded, so the accelerating field does not change sign occurs at

$$C = 0 \Rightarrow R_b^4 = 8 \Lambda_0 r_t^2$$ ← Call this optimum beam loading for a flat top

Integrating the E_z differential eqn [Eqn 22(a) & 22(b)] allows us to obtain an expression for $r_b(\xi)$ using the incomplete integrals of first & second kind, E & F respectively (see Appendix of Tzoufras 2009)

$$\frac{\xi}{R_b} = 2E\left[\arccos\left(\frac{r_b}{R_b}\right) \Big| \frac{1}{2}\right] - F\left[\arccos\left(\frac{r_b}{R_b}\right) \Big| \frac{1}{2}\right] \quad \ldots (24)$$

Expressions for max beam loading $(C=0)$ within the beam $\xi_t \leq \xi \leq \xi_F$ are

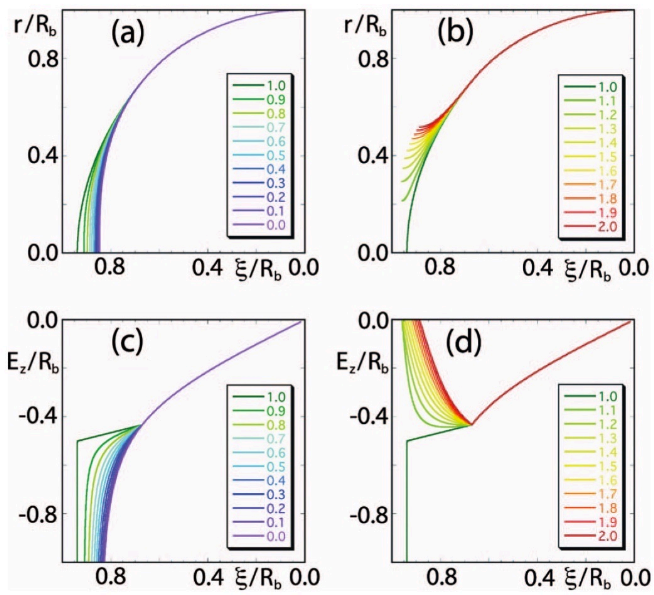
$$\begin{cases} E_z = -\frac{1}{4}(\xi - \xi_t) + E_z(\xi_t) & \ldots (25) \\ 8\Lambda_0 = r_b^2 + \frac{1}{2}\left(\xi - \xi_t + \sqrt{2}\sqrt{8\Lambda_0 - r_t^2}\right)^2 & \ldots (26) \end{cases}$$

Solutions for bunches with $C > 0$, $R_b^2/8r_t^2 > \Lambda_0$, i.e. Low charge per unit length for $\xi_t = 0.67 R_b$ shown in Fig. (a) & (c) below (4a & 4c in Tzoufras 2009)

As the charge per unit length increases (starting from Λ_0=0) the bubble elongates (Fig. a) and the wakefield within the bunch flattens (Fig. c). For the largest bubble, for which C=0, the slope within the bunch is -1/4 as in Eq. 3

For $C < 0$, $\Lambda_0 > R_b^4/8r_t^2$, i.e. high charge, solutions are plotted for $\xi_t = 0.67 R_b$ in Fig b & d

As charge per unit length increases the electrons in the sheath do not return on axis, but instead turn around after they reach a minimum value ($r_m > 0$) at which point, the wakefield crosses zero.



<u>Beam loading for a constant wakefield</u>

The condition we seek is to have a current profile such that

$$E_z(r_b \leq r_t) = \frac{1}{2} r_b \left(\frac{dr_b}{d\xi}\right)_{r_b = r_t} \approx \text{constant} = -E_t \quad \cdots \quad (27)$$

For the part before the trailing beam, the field is the same as the previous section:

$$E_z(\xi) = -\frac{r_b}{2\sqrt{2}}\sqrt{\frac{R_b^4}{r_b^4} - 1} \quad \cdots \quad (28)$$

$$\Rightarrow E_t = E_z(\xi_t)$$

$$\therefore E_z(\xi > \xi_t) = E_t = -\frac{r_t}{2\sqrt{2}}\sqrt{\frac{R_b^4}{r_t^4} - 1} \quad \cdots \quad (29)$$

The shape of the bubble is described by parabola:

$$\frac{1}{2} r_b r'_b = E_t$$

Boundary condition: $\xi = \xi_t$, $r_b = r_t$

$$\Rightarrow r_b^2 = r_t^2 - 4E_t(\xi - \xi_t) \quad \cdots \quad (30)$$

Once we have r_b, we can solve for λ directly from Eqn 11:

$$(11) \Rightarrow r_b \frac{d^2 r_b}{d\xi^2} + 2\left(\frac{dr_b}{d\xi}\right)^2 + 1 = \frac{4\lambda(\xi)}{r_b^2}$$

$$\lambda(\xi) = E_t^2 + \frac{r_b^2}{4} = \frac{R_b^4 + r_t^4}{8 r_t^2} - \sqrt{\frac{R_b^4 - r_t^4}{8 r_t^2}} (\xi - \xi_t) \quad \ldots (31)$$

This expression can be written only in terms of Et instead of rt by solving r_t^2 in terms of Et from equation 29

$$\lambda(\xi) = \sqrt{E_t^4 + \frac{R_b^4}{2^4}} - E_t (\xi - \xi_t) \quad \ldots (32)$$

This is the equation for a trapezoid.

<u>Maximum total charge</u>

We can calculate the maximum total charge that corresponds to loading the fields to Et by assuming that the charge extends all the way tot the back of the bubble, where the sheath reaches the ξ axis.

use $\qquad r_b^2 = r_t^2 - 4 E_t (\xi - \xi_t)$

$$r_b \to 0 \Rightarrow \xi - \xi_t = \Delta\xi_{tr} = \frac{r_t^2}{4E_t} \quad \ldots (33)$$

Average charge per unit length for the trapezoidal bunch:

$$\langle \lambda \rangle_{\Delta tr} = \frac{\lambda(\xi_t + \Delta\xi_{tr}) + \lambda(\xi_t)}{2} = \frac{R_b^4}{8 r_t^2} = \Lambda_0 \quad \ldots (34)$$

from the case of $C=0$ from previous section

$$Q_{tr} = 2\pi \Lambda_0 \Delta\xi_{tr} = \boxed{\frac{\pi}{16} \frac{R_b^4}{E_t}} \quad \ldots (35)$$

<u>Simulation results</u>

Here, we look at the simulation results using the parameters described in Tzoufras 2009:

$$n_b(r, \xi) = \frac{N_b}{(2\pi)^{3/2} \sigma_r^2 \sigma_z} e^{-r^2/2\sigma_r^2} e^{-(\xi - \xi_c)^2/2\sigma_z^2} \quad \ldots (37)$$

$\sigma_r = 0.5\ c/\omega_p$, $N_b = 139\,(c/\omega_p)^3$

$\sigma_z = 1.414\ c/\omega_p$, $\xi_c = 10\ c/\omega_p$ $\Big\} \Rightarrow k_p R_b = 5$

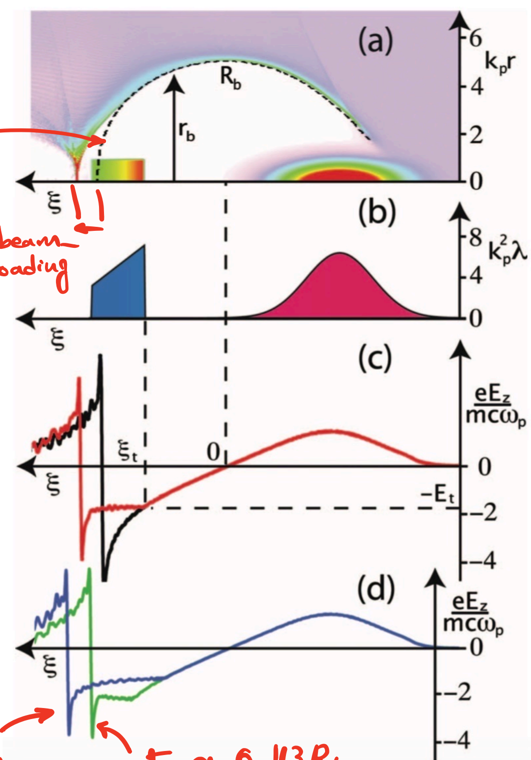
optimal trapezoid Loaded at $\xi_t = 18.2\ c/\omega_p$

observed from simulations $\longrightarrow$ $E_t = 0.35\, R_b = 1.75\ c/\omega_p$

Calculated
$$\begin{cases} Q_{tr} \simeq 70/k_p^3 & \text{(loaded charge)} \\ k_p^2\lambda = 7.0 - 1.75\times(\xi - 18.2) & \text{(optimal beam density)} \\ \Delta\xi_{tr} = [\lambda(\xi_t) - E_t^2]/E_t = 2.2\ c/\omega_p & \end{cases}$$

r_b without beam loading

elongation due to beam loading



$E_t \approx 0.25 R_b = 1.3\ c/\omega_p$

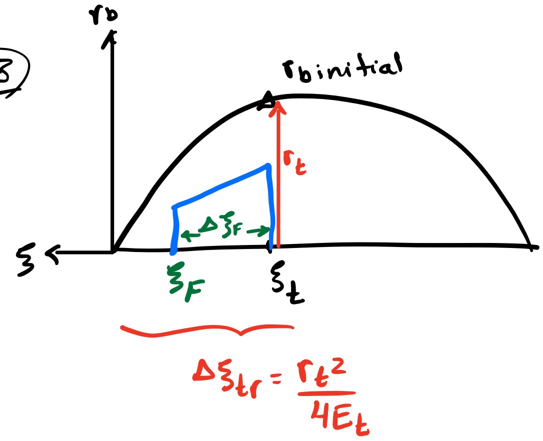
$E_t \approx 0.43 R_b = 2.2\ c/\omega_p$

For these latter cases the wakefield is not as flat. The reason for this is that kpRb is not large enough for Eq. 11 to be completely accurate. This illustrates the size of errors that may result if kpRb is not large enough. If the charge of the bunch is increased/ decreased slightly for blue/green cases, the wakes can then be made to be more flat. For very large blowout radii the differences between theory and simulation are negligible.

Beam loading efficiency of nonlinear beam loading

Let us start by assuming that beam loading terminated at some ξ_F, such that

$$\Delta\xi_F = \xi_F - \xi_t < \xi_{tr} \quad \cdots (38)$$



For $\xi > \xi_F$, $\Lambda = 0$,

so the wakefield is described by the equation 22a with a new peak blowout radius $\tilde{R}_b$

$$E_z = -\frac{r_b}{2\sqrt{2}} \sqrt{\frac{\tilde{R}_b^4}{r_b^4} - 1}$$

By solving for this constant ($\tilde{R}_b$), we obtain the trajectory of the innermost electron behind the bunch, which allows us to determine the energy remaining in the wake. This constant can be found by using the continuity of the field at ξ_F:

$$\begin{cases} E_F = E_z(r_F) = \dfrac{-r_F}{2\sqrt{2}} \sqrt{\dfrac{\tilde{R}_b^4}{r_F^4} - 1} = E_t \\ r_F^2 = r_t^2 - 4E_t(\xi_F - \xi_t) \\ E_t = \dfrac{-r_t}{2\sqrt{2}} \sqrt{\dfrac{R_b^4}{r_t^4} - 1} \end{cases}$$

since E_z field is flattened starting at ξ_t

$$\Rightarrow \boxed{\tilde{R}_b^4 = R_b^4 \left(\frac{r_F^4}{R_b^4} + \frac{r_F^2}{r_t^2} - \frac{r_t^2 r_F^2}{R_b^4} \right)} \quad \cdots (39)$$

From the discussions of energy consideration above, we know that

$$I_0 = \pi/16 \, R_b^4$$

represents the power of energy exchange between the field of the bubble and the particle bunch. In other words, for the bubble before beamloading, I_0 represents the energy given to the wakefield by the driver per unit time. Behind the trailing bunch, I_0 drops to

$$I_0 = \pi/16 \, \tilde{R}_b^4 ,$$

which is the energy left in the wakefield. Therefore, the efficiency of beam loading, being the efficiency with which the trailing beam extracts energy is given by

$$\eta_b = 1 - \left(\frac{\tilde{R}_b}{R_b}\right)^4 \quad \ldots (40)$$

This equation in principle holds for any bunch shape. For a flattened field, we can use equation 7 to find the efficiency of beam loading:

$$\eta_b = 1 - \left(\frac{r_F^4}{R_b^4} + \frac{r_F^2}{r_t^2} - \frac{r_t^2 r_F^2}{R_b^4}\right) \quad \ldots (41)$$

Using the expression for Q_{tr} (Equation 35),

$$Q_{tr} = \frac{\pi R_b^4}{16 E_t} \quad \text{(maximum charge)}$$

and an equivalent expression for Q_F (bunch ends at ξ_F), we get

$$\boxed{\eta_b = \frac{Q_F}{Q_{tr}}} \quad \ldots (42)$$

Note : as $\Delta\xi_F \to \Delta\xi_{tr}$

$$Q_F \to Q_{tr} \Rightarrow \eta_b \to 100\%$$

This equation, which is for a trapzoidal beam can be generalized for arbitrary electron bunches as the ratio of rate of energy gain by the arbitrary bunch over the rate of energy gain by the optimal trapezoidal bunch:

$$\eta = \frac{\int E \, dq}{E_t Q_{tr}} = \frac{2\pi \int_0^{\xi_F} E_z(\xi) \lambda(\xi) \, d\xi}{E_t Q_{tr}} = \frac{I_0(F) - I_0(0)}{I_0(t) - I_0(0)} \quad \ldots (43)$$

e.g. for an optimally loaded flat top extended all the way to the back of bubble:

$$2\pi \int_0^{\xi_F} E_z(\xi)\,\lambda(\xi)\,d\xi = 2\pi \Lambda_0\, \Delta\xi \,\langle E_z \rangle_{\Delta\xi} \quad \ldots (44)$$

For the optimal case $\left(C = 0 \Rightarrow R_b^4 = 8\Lambda_0 r_t^2\right)$ & expressions for E_z & $\Delta\xi$ in this case (see Tzoufras Appendix),

$$Q\langle E_z \rangle_{\Delta\xi} = 2\pi \Lambda_0 \frac{r_t^2}{4} = \frac{\pi R_b^4}{16} \quad \ldots (45)$$

This is the same expression as $Q_{tr} E_t$!

Therefore, a flat top beam in the optimal condition (C=0) also leads to near 100% efficiency. For smaller charge per unit length (C>0), the plasma electrons reach the ξ axis quickly and with large transverse velocity. As a result they overshoot and continue to oscillate. For higher charge per unit length (C<0), the innermost electron velocity changes sign and so they get some potential energy from the trailing beam. In either case, some energy is left in the plasma electrons behind the driver, which reduces efficiency.

<u>Comparison of beam loading in linear and nonlinear regimes</u>
We can compare the linear and nonlinear beam loading by comparing the energy absorbed per unit length on both cases in unnormalized units:

$$\frac{Q_{tr} E_t}{mc\omega_p} = \frac{\pi}{16}\left(k_p R_b\right)^4 \approx 10 \quad \text{for } k_p R_b \sim 3$$

For the linear regime, take $A = A_{eff} = \frac{c^2}{\omega_p^2}$ for a narrow beam.

$$\frac{Q_{tr} E_t}{mc\omega_p} = \frac{1}{2} \underbrace{\sin^2 \xi_0}_{\text{order } 1} \underbrace{\left(\frac{\delta n}{n}\right)^2}$$

Linear regime: $\delta n/n \ll 1$, e.g. $\frac{\delta n}{n} \sim 0.1$
$\Rightarrow \left(\frac{\delta n}{n}\right)^2 \sim 0.01$

In the blowout regime for a moderate radius, $k_p R_b \sim 3$, total accelerating force is orders of magnitude larger than that in the linear regime.

Physical picture:

Total energy per unit length $\propto$ $E^2 A$ ← wake cross section

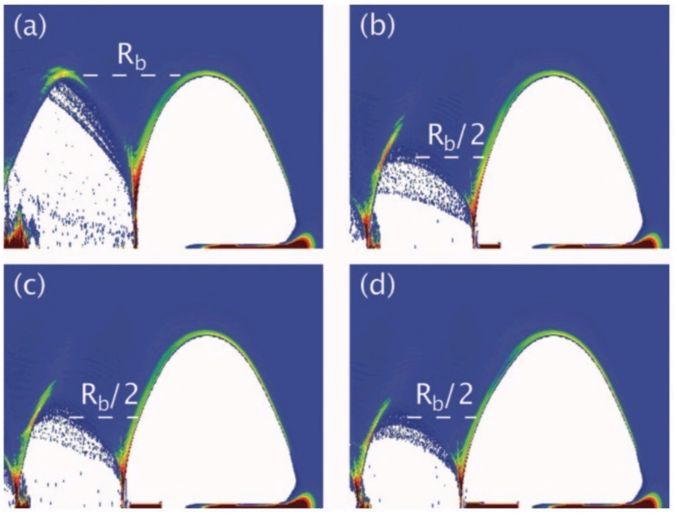
In linear regime E behind the trailing beam is reduced while A stays the same

In nonlinear regime both E^2 & A scale as R_b^2 $\Rightarrow$ Trailing beam reduces both, leading to more efficient absorption.

(a): no beam loading

(b): optimal trapezoidal bunch

(c), (d): trapezoidal bunch in different locations



The size of the blowout behind the beam loading is dropped to around 1/2.
So the efficiency is

$$\eta_b = 1 - \left(\tfrac{1}{2}\right)^4 = 93.75\,\%$$

This is an extremely efficient process by accelerator physics standards!

Note that the similar Rb, and therefore efficiency for cases (b-d) confirms the theoretical prediction that, in contrast to linear regime, the efficiency is independent of the accelerating gradient.

<u>Note 1: on experiencing dephasing</u>

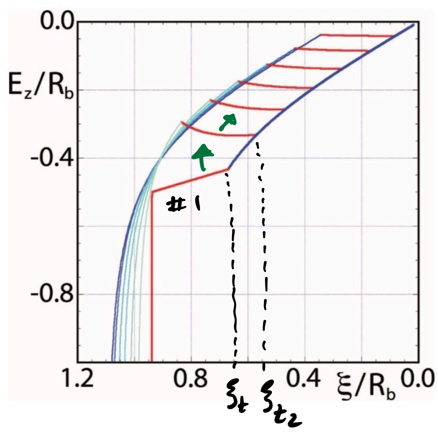
If the wake is driven by an ultrarelativistic electron beam the accelerating electron bunch can be assumed to be phase locked with the wake:

$$\gamma^{\text{driver}} \sim 10^4, \quad \text{no phase slippage}$$

In this case a bunch with optimal trapezoidal profile conserves its energy spread throughout the acceleration process.

For a laser driver, however, the accelerating electron bunch moves faster than the wake and it samples multiple phases of the accelerating field. As a result, even if the current profile of the bunch is chosen so that the wake is initially flat, this stops being the case as soon as the bunch moves to a different phase.

To study this issue we use the theoretical solutions derived for flat-top bunch:
Assume an initially optimal bunch



$$C(\xi_t)=0 \xrightarrow{\text{curve \#1 on figure}}$$

As the trailing bunch outruns the wake, the trailing beam density becomes too large for optimum loading, I.e.

$$\xi_{t_2} < \xi_t$$

$$r_{t_2} > r_{t_1}$$

$$C(\xi_{t_2}) < C(\xi_t) = 0$$

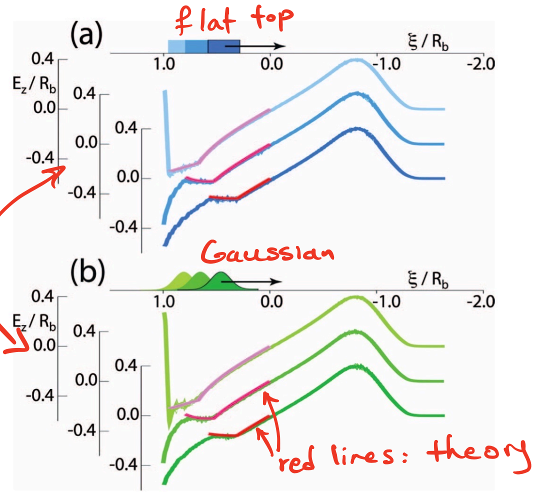
As the bunch keeps approaching the center of the bubble, C continues to decrease. We can see from the figure above that when C<0, there is a minimum for Ez, which occurs within the electron bunch. As a result, there is a region around this minimum for which $dE_z/d\xi = 0$. Therefore, while the energy spread for the whole bunch is not technically preserved, the energy spread around this minimum is roughly preserved.

Moreover, as the bunch moves further forward, this plateau becomes longer and as a result the entire bunch is in a region with nearly flat Ez. This can be seen in the figure above as well.

Therefore, in this situation, while the wake initially has a negative slope, as the bunch moves forward with respect to the wake, the wake becomes flat at the front and (in some places) has a small positive slope at the back.

Note 2: on Gaussian beam profiles

In 3D nonlinear regime, it appears that the behavior of a Gaussian electron bunch mimics that of the flat-top bunch, given



$K_pR_b \sim 10$

width of flat top $\rightarrow \Delta\xi = 2\sqrt{2}\,\sigma_z$,

$$\text{where } N_b(z) = \frac{N_b}{\sqrt{2\pi}\,\sigma_z}\, e^{-z^2/2\sigma_z^2}$$

This indicates that Gaussian bunches can be accelerated in the wake of a laser/beam with very high efficiency and nearly constant wakefield. Additionally, if such bunches are extracted before they reach the center of the bubble, their quality will not be significantly affected.

For the simulations in previous page, the Gaussian beams are centered at a distance of $\sqrt{2}\,\sigma_z$ from the edge of the corresponding flat-top profiles.

Numerical Example (from Tzoufras, PRL, 101, 145002, 2008)

For a bi-Gaussian beam with $k_p\sigma_z \sim 1$ & $k_p\sigma_r \ll 1$,

$$K_pR_b = 2\sqrt{\Lambda} = 2\sqrt{\frac{n_b}{n_0}(k_p\sigma_r)^2}$$ (for a matched Laser, use $K_pR_b \simeq 2\sqrt{a_0}$)

e.g. electron drive beam

Let $N = 3\times10^{10}$ electrons

$\sigma_r \ll \sigma_z = 16.8\,\mu m$

$n_p = 10^{17}\ cm^{-3}$

Laser

$P = 200\,TW$

$n_p = 1.2\times10^{18}\ cm^{-3}$

$\rightarrow K_pR_b = 4 \leftarrow$

$$\frac{eE_t}{mc\omega_p} = \frac{1}{2}K_pR_b \simeq 2 \Rightarrow Q_t = 1.9\,nC \quad \text{e-beam}$$

max field

or

$Q_t = 0.55\,nC$

Dechirping

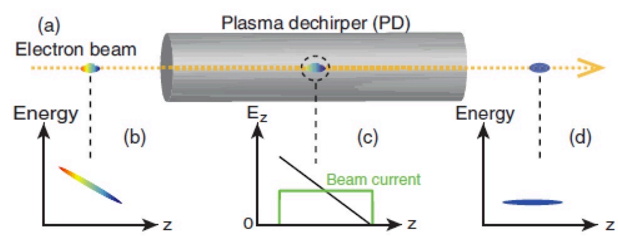
Over the last decade since Tzoufras proposed the ideal beam loading with a trapezoidal-shaped beam, the creation of such a beam has remained an unsolved research problem. Another solution has emerged based on the fact that the beam loading of a flat-top or a Gaussian beam load results in a linear accelerating field. The electrons accelerated thus have a spatial “linear chirp” at the end of their acceleration length.

So rather than creating a situation for a flat electric field inside the bubble, several simulation and experimental groups have investigated the idea of dechirping, I.e. removing the linear chirp after the beam goes through its acceleration length. Below I will describe two such ideas commonly discussed in the community

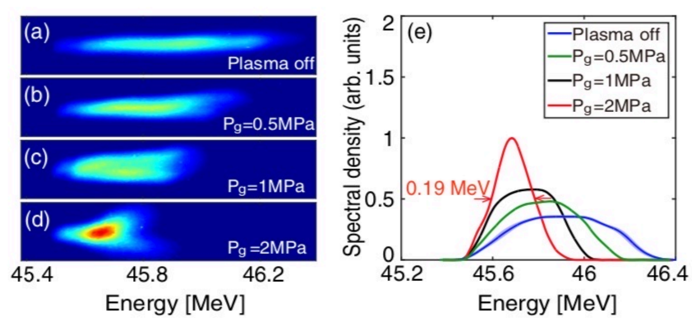
1. Sending the resulting beam through low density plasma (see e.g. Wu, et al., PRL, 204804, 2019)

The initially chirped beam is sent through a low density plasma and if the parameters of the plasma are chosen properly, the self fields generated in the plasma produce the opposite chirp resulting in a beam with almost no energy spread. Simulations suggest a 10x reduction in energy spread down to 0.1% is possible.

Schematic of the idea with the self fields in the plasma counteracting the initial energy spread.

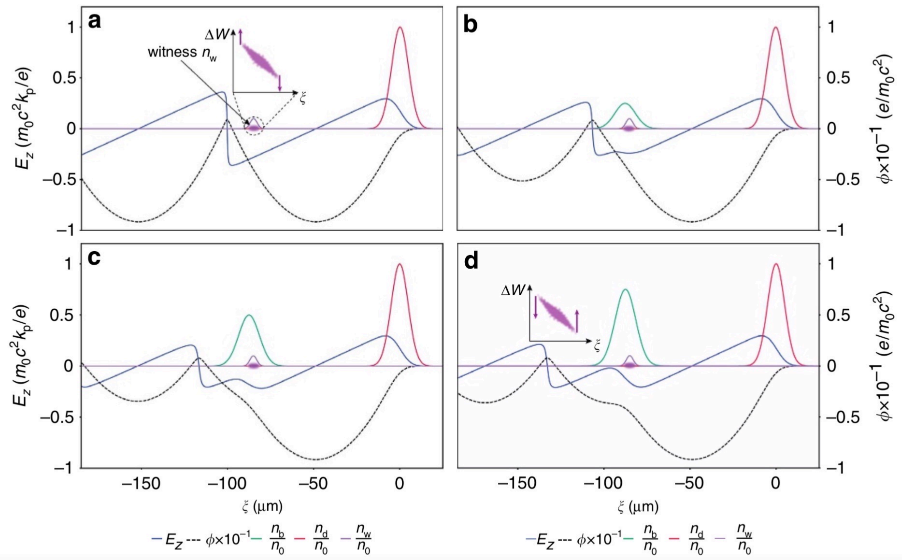


Experiments snd stimulants show that the energy spread can be significantly reduced.



2. Proper overloading of the wake by an escort beam (see e.g. Manahan, Et al. Nature Communications 8, 15705, 2017)

The idea in this paper is to use a Gaussian and accelerate it until it has reached the desire energy. At that point, a second bunch can be injected to co-propagate with the initial beam, with its position and charge set such that C<0 in the position of the original beam. The opposite slope that is generated then will remove the energy spread from the initial beam.



The figure above shows how the slope of the accelerating field changes for different charges in the escort beam (all beams have Gaussian profiles). The idea is to transition from a case where C=0, to where C<0.