Comprehensive Examination

Department of Physics and Astronomy

Stony Brook University

Spring 2021 (in 4 separate parts: CM, EM, QM, SM)

General Instructions:

Three problems are given. If you take this exam as a placement exam, you must work on all three problems. If you take the exam as a qualifying exam, you must work on two problems (if you work on all three problems, only the two problems with the highest scores will be counted).

Each problem counts for 20 points, and the solution should typically take approximately one hour.

Use one exam book for each problem, and label it carefully with the problem topic and number and your ID number.

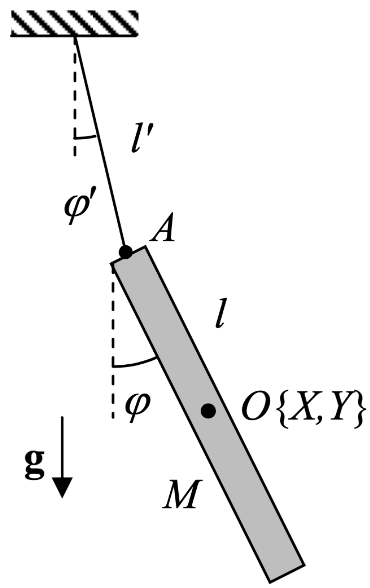
Write your ID number (not your name!) on each exam booklet.

You may use, one sheet (front and back side) of handwritten notes and, with the proctor's approval, a foreign-language dictionary. **No other materials may be used**.

Classical Mechanics 1

A bar on a thread

A thin uniform bar of mass M and length l is hung on a light thread of length l' in a gravitational field with acceleration, g (see above). Take the origin of the coordinate system to be the point at which the thread attaches to the ceiling on the top. The center of mass of the bar is at O (whose Cartesian coordinates are X and Y). Let I be the moment of inertia of the bar relative to its center of mass O, i.e., $I = \frac{1}{12}Ml^2$. Let ω be the instantaneous angular velocity of the bar's rotation in the plane of the drawing.



(a) **[5 points]** Find the equations of motion of the system. A useful set of generalized coordinates are the two angles φ and φ' as indicated in the figure. (You may wish to use the notation $I_A = I + \frac{1}{4}Ml^2 = \frac{1}{3}Ml^2$, where I_A is the moment of inertia around point A.)

(b) **[2 points]** Linearize the equations of motion near their fixed point of $\varphi = \varphi' = 0$ (i.e., keep only terms linear in φ, φ', $\dot{\varphi}$, $\dot{\varphi}'$, $\ddot{\varphi}$, and $\ddot{\varphi}'$ as needed).

(c) **[6 points]** Find the eigenfrequencies of small oscillations near the equilibrium. Write your answer in terms of $\Omega^2 \equiv \frac{3g}{2l}$ and $\Omega'^2 \equiv \frac{g}{l'}$.

(d) **[3 points]** Discuss the motion in the limits (i) $l \ll l'$ and (ii) $l' \ll l$. In particular, find the oscillation frequencies and discuss the physical significance of the two limits. You may draw a sketch if helpful.

(e) **[4 points]** Find the eigenmodes for the particular case $l = l'$. Sketch the two oscillation modes in this limit.

Solutions

1. The Lagrangian of the system is

$$L = T - U = \frac{1}{2}M(\dot{X}^2 + \dot{Y}^2) + \frac{1}{2}I\omega^2 - MgY\,.$$

We can express X and Y in terms of the generalized coordinates φ and φ' as

$$X = l'\sin\varphi' + \frac{1}{2}l\sin\varphi, \quad Y = -l'\cos\varphi' - \frac{1}{2}l\cos\varphi, \quad \omega = \dot{\varphi}\,.$$

With these expressions, the Lagrangian becomes

$$L = \frac{1}{2}M\left(l'^2\dot{\varphi}'^2 + \frac{1}{4}l^2\dot{\varphi}^2 + ll'\cos(\varphi - \varphi')\dot{\varphi}\dot{\varphi}'\right) + \frac{1}{2}I\dot{\varphi}^2 + Mg\left(l'\cos\varphi' + \frac{1}{2}l\cos\varphi\right)\,.$$

We can now use the Euler-Lagrange equations to find the following equations of motion for φ and φ':

$$I_A\ddot{\varphi} + \frac{1}{2}Mll'\cos(\varphi - \varphi')\ddot{\varphi}' + \frac{1}{2}Mll'\sin(\varphi - \varphi')\dot{\varphi}'^2 + \frac{1}{2}Mgl\sin\varphi = 0 \tag{1}$$

$$Ml'^2\ddot{\varphi}' + \frac{1}{2}Mll'\cos(\varphi - \varphi')\ddot{\varphi} - \frac{1}{2}Mll'\sin(\varphi - \varphi')\dot{\varphi}^2 + Mgl'\sin\varphi' = 0\,. \tag{2}$$

2. The equations are simplified by linearizing them near the fixed point $\varphi = \varphi' = 0$, which describes the equilibrium position:

$$I_A\ddot{\varphi} + \frac{1}{2}Mll'\ddot{\varphi}' + \frac{1}{2}Mgl\varphi = 0 \tag{3}$$

$$Ml'^2\ddot{\varphi}' + \frac{1}{2}Mll'\ddot{\varphi} + Mgl'\varphi' = 0\,. \tag{4}$$

3. We look for solutions to the equations of motion of the form

$$\varphi = ae^{-i\omega t}, \quad \varphi' = a'e^{-i\omega t}\,. \tag{5}$$

Inserting Eq. (5) into Eq. (3) and Eq. (4), we get a system of linear equations for the oscillation amplitudes a and a':

$$\left(\frac{1}{2}Mgl - \omega^2 I_A\right)a - \frac{1}{2}\omega^2 Mll'a' = 0 \tag{6}$$

$$-\frac{1}{2}\omega^2 Mll'a + \left(Mgl' - \omega^2 Ml'^2\right)a' = 0\,, \tag{7}$$

or, in terms of Ω and Ω',

$$\left(\Omega^2 - \omega^2\right)a - \frac{\Omega^2}{\Omega'^2}\omega^2 a' = 0 \tag{8}$$

$$-\frac{3\Omega'^2}{4\Omega^2}\omega^2 a + \left(\Omega'^2 - \omega^2\right)a' = 0\,. \tag{9}$$

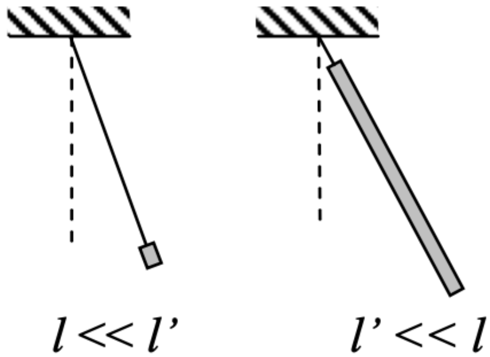


Figure 1: Physical significance of the two limits $l \ll l'$ and $l' \ll l$.

Here, Ω is the oscillation frequency of the bar suspended by the point A (with $\varphi' = 0$), while Ω' is the oscillation frequency of the system if all the bar's mass is concentrated at point A. We determine the frequencies of the small oscillations by requiring the determinant of these equations written in matrix form to be zero:

$$\begin{vmatrix} \Omega^2 - \omega^2 & -\frac{\Omega^2}{\Omega'^2}\omega^2 \\ -\frac{3\Omega'^2}{4\Omega^2}\omega^2 & \Omega'^2 - \omega^2 \end{vmatrix} = \frac{1}{4}\left(\omega^4 - 4(\Omega^2 + \Omega'^2)\omega^2 + 4\Omega^2\Omega'^2\right) = 0\,. \tag{10}$$

The two solutions of this quadratic equation for ω^2 are

$$\omega_\pm^2 = 2\left(\Omega^2 + \Omega'^2\right) \pm 2\sqrt{(\Omega^2 + \Omega'^2)^2 - \Omega^2\Omega'^2}\,. \tag{11}$$

4. (i) For $l \ll l'$, the bar is shrunk to a point; in this case, $\Omega \gg \Omega'$ and $\omega_- \simeq \Omega'$. (ii) For $l' \ll l$, the bar is essentially attached to the ceiling at point A; in this case, $\Omega' \gg \Omega$ and $\omega_- \simeq \Omega$. See Fig. 1.

5. For $l = l'$, $\Omega'^2 = \frac{2}{3}\Omega^2$, so that

$$\omega_\pm^2 = \frac{2}{3}\left(5 \pm \sqrt{19}\right)\Omega^2\,, \tag{12}$$

i.e., $\omega_+ \simeq 2.496\Omega$ and $\omega_- \simeq 0.654\Omega$. Inserting these values, one by one, into Eqns. (8) and (9), we find

$$\left(\frac{a'}{a}\right)_+ = -0.560 \qquad \left(\frac{a'}{a}\right)_- = -0.893\,. \tag{13}$$

The oscillations look as in Fig. 2.

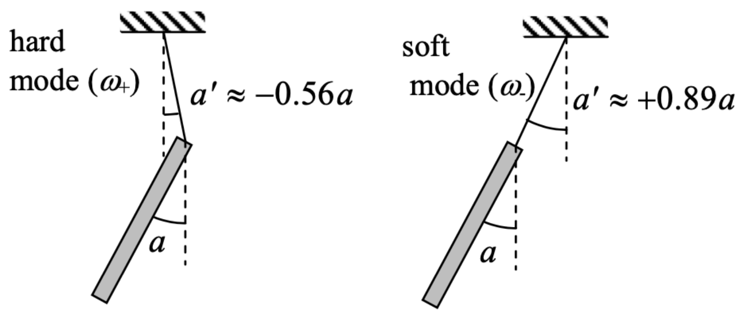
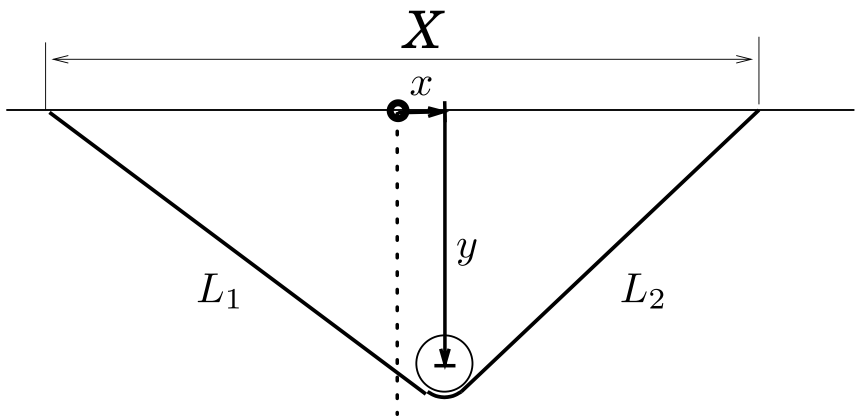


Figure 2: The two distinct oscillation modes for $l = l'$.

Classical Mechanics 2

Rolling of a spool

A wheel is allowed to roll on a massless rope of the length L without slipping. Both ends of the rope are fixed to the ceiling as shown below. The wheel is a homogeneous disk of the mass M and radius R. Assume that X is of order L, but $R \ll L$.



(a) (4 points) Recall that an ellipse is the locus of points such that the sum of distances $L_1 + L_2 = L$ is constant (the case here). The Cartesian form of an ellipse reads

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \,. \tag{14}$$

where x and y are measured from the origin (see above). Determine the Cartesian parameters a and b of in terms of the variables L and X. Show that the L_1 and L_2 are related to the x displacement of the disk for $R \ll L$ (see figure).

$$L_1 = \frac{L}{2} + \frac{X}{L}x \,, \tag{15}$$

$$L_2 = \frac{L}{2} - \frac{X}{L}x \,. \tag{16}$$

Do not assume that x is small.

(b) (7 points) Write down the Lagrangian for the system. Do not assume x is small. Take x as a generalized coordinate.

(c) (4 points) Determine frequency of small oscillations near equilibrium.

(d) (5 points) Now consider what happens if the separation X is slowly increased. If the initial separation is X_0 and the initial amplitude of small oscillations is A_0, determine how the final oscillation amplitude depends on X/L. What happens for $X \simeq L$?

Solution:

(a) Let L_1 and L_2 be the lengths of the two parts of the rope situated to the left and to the right from the wheel. If $R \ll L$ we can write $L_1 + L_2 = L$. Also, let x and y be the coordinates of the center of the wheel, with the origin of the coordinates located exactly in the middle between the two points where the rope is attached to the ceiling, see Figure below. By construction then, the position (x, y) is constrained to lay on the ellipse.

The Cartesian form is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \,. \tag{17}$$

We can set $x = 0$ and then $y^2 = b^2$ with $b = \sqrt{(L/2)^2 - (X/2)^2}$ being the semi-minor axis of the ellipse. Similarly, setting $y = 0$, then $x = a$, with $a = (L_1 + L_2)/2 = L/2$ the semi-major axis of the ellipse. (Draw a picture if confused). We find then the relation

$$y^2 = \frac{L^2 - X^2}{L^2}\left(\frac{L^2}{4} - x^2\right) . \tag{18}$$

To relate L_1 and L_2 to x and y we have

$$L_1^2 =(X/2 + x)^2 + y^2 \,, \tag{19}$$
$$L_2^2 =(X/2 - x)^2 + y^2 \,. \tag{20}$$

So, subtracting the two equations we have

$$L_1^2 - L_2^2 = 2xX \,. \tag{21}$$

Writing $L_2 = L - L_1$ we find

$$2LL_1 - L^2 = 2xX \,, \tag{22}$$

and then it it is easy to find the result quoted in the problem statement

$$L_1 =\frac{L}{2} + \frac{X}{L}x \,, \tag{23}$$
$$L_2 =\frac{L}{2} - \frac{X}{L}x \,. \tag{24}$$

(b) To write down the Lagrangian of the configuration we will need to relate the angle of rotation ϕ to the displacement x. The rotational configuration of the wheel is described by a single angle ϕ. If the radius of the wheel R is small, the angle relates to the lengths L_1 (or L_2) in a simple way. The distance along the string that the disk has rolled as x is increased from zero is

$$R\phi = L_1 - L/2 \,, \tag{25}$$

where we have assumed $R \ll L$, and taken positive ϕ as clockwise. Thus ϕ is a simple (linear) function of x,

$$\phi = \frac{X}{LR}x \,. \tag{26}$$

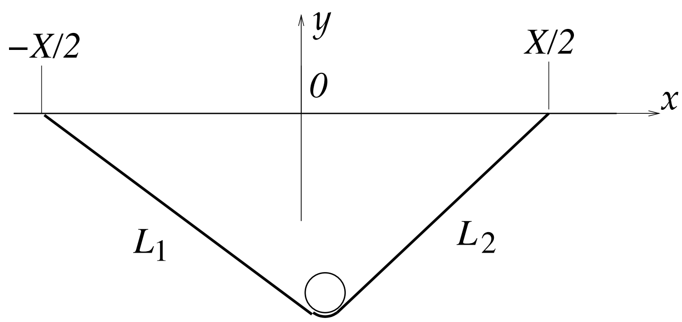


Figure 3: A wheel on a massless rope.

The kinetic energy of the wheel is

$$K = \frac{M}{2}(\dot{x}^2 + \dot{y}^2) + \frac{I}{2}\omega^2, \tag{27}$$

where $I = MR^2/2$ is the moment inertia (the wheel is assumed to be a homogeneous disk), and $\omega = \dot{\phi}$ is the angular velocity of the wheel. Taking into account the constraints in (18) and (26), we have finally

$$L = \frac{M}{2}\left(\frac{L^4 - 4X^2x^2}{L^2(L^2 - 4x^2)} + \frac{X^2}{2L^2}\right)\dot{x}^2 + Mg\sqrt{\frac{L^2 - X^2}{L^2}\left(\frac{L^2}{4} - x^2\right)}\,. \tag{28}$$

(c) The point of static equilibrium is $x = 0$. Expanding near this point, and keeping only quadratic terms, we have

$$L = \frac{M}{2}\left(1 + \frac{X^2}{2L^2}\right)\dot{x}^2 - Mg\frac{\sqrt{L^2 - X^2}}{L^2}x^2 + \text{const}\,, \tag{29}$$

$$= \frac{1}{2}m(X)\dot{x}^2 - \frac{1}{2}m(X)\Omega^2x^2\,. \tag{30}$$

where we have defined the frequency and effective mass

$$\Omega^2 = \frac{4g\sqrt{L^2 - X^2}}{2L^2 + X^2}\,, \tag{31}$$

$$m(X) = M\left(1 + \frac{X^2}{2L^2}\right)\,. \tag{32}$$

Ω determines the frequency of small oscillations near the equilibrium point.

(d) From the general theory of adiabatic invariants, the energy per frequency is constant under a slow change of a parameter

$$I \equiv \frac{E}{\omega} = \text{const}\,, \tag{33}$$

which is a consequence of the Louiville theorem. Here

$$E = \frac{1}{2} m(X) \Omega^2 A^2 \,. \tag{34}$$

is the first integral of the one dimensional motion.

Let E_0, Ω_0, X_0, A_0, m_0 be the initial values of the parameters, and E_f, Ω_f , X_f, A_f , m_f be the final values of these parameters. The adiabatic condition gives

$$m_f \Omega_f A_f^2 = m_0 \Omega_0 A_0^2 \,. \tag{35}$$

Solving for A_f we have

$$A_f = A_0 \sqrt{\frac{m_0 \Omega_0}{m_f \Omega_f}} \,. \tag{36}$$

Substituting $X_0 = L/2$ and $X_f = L/4$, we find after minor algebra

$$m\Omega = \mathrm{const}(1 - u^2)^{1/4}(1 + u^2/2)^{1/2} \,, \tag{37}$$

where $u \equiv X/L$, and thus

$$A_f = \frac{\mathrm{const}}{(1 - u^2)^{1/8}(1 + u^2/2)^{1/4}} \,. \tag{38}$$

The amplitude grows as $(1 - X/L)^{-1/8}$ for $X \simeq L$.

Classical Mechanics 3

Dissipation from an external field

A particle of mass m moves in one dimension, parametrized by coordinate q, subject to a potential energy $V(q)$ and to a damping force

$$F_{\text{friction}} = -2m\gamma\dot{q}\,.$$

(a) (4 points) Find a Lagrangian that gives the correct equation of motion for the particle. *Hint:* consider a Lagrangian of the form $L(q,\dot{q},t) = f(t)\,L_0(q,\dot{q})$.

(b) (2 points) Find the corresponding Hamiltonian $H(P,Q,t)$.

(c) (8 points) Assume that the potential is harmonic,

$$V(q) = \frac{1}{2}m\omega^2 q^2\,.$$

(To slightly simplify the algebra, henceforth you may set $m \equiv 1$ and $\omega \equiv 1$ by a choice of units). Find a canonical transformation

$$Q = Q(q,p,t)\,, \quad P = P(q,p,t)$$

such that the transformed Hamiltonian $K(Q,P)$ does not depend explicitly on time.

(d) (6 points) Write the Hamiltonian equations of motion that follows from $K(Q,P)$. Find the general solution for $Q(t)$ in the underdamped case $\gamma < \omega \equiv 1$. Transform back to the original coordinate q and verify that this gives the familiar solution for an underdamped harmonic oscillator.

Solution:

(a) We are looking for a Lagrangian that yields the equation of motion

$$m\ddot{q} + 2m\gamma\dot{q} + V'(q) = 0\,. \tag{1}$$

For $\gamma = 0$ the Lagrangian is clearly

$$L_{\gamma=0}(q,\dot{q}) = T - V = \frac{1}{2}m\dot{q}^2 - V(q)\,. \tag{2}$$

This suggests to refine the ansatz in the hint by taking $L_0 = L_{\gamma=0}$,

$$L(q,\dot{q},t) = f(t)\left(\frac{1}{2}m\dot{q}^2 - V(q)\right)\,. \tag{3}$$

The Euler-Lagrange equations of motion that follow from this ansatz are

$$0 = -\frac{\partial L}{\partial q} + \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{q}}\right) = f(t)V'(q) + f(t)m\ddot{q} + f'(t)m\dot{q}\,. \tag{4}$$

Comparing with (1), we identify $f'/f = 2\gamma$, which gives $f(t) = c\,e^{2\gamma t}$, where c is an arbitrary integration constant that gives an overall rescaling of L and may be set to 1. So the answer is

$$L(q,\dot{q},t) = e^{2\gamma t}\left(\frac{1}{2}m\dot{q}^2 - V(q)\right)\,. \tag{5}$$

(b) The canonical momentum is

$$p \equiv \frac{\partial L}{\partial \dot{q}} = e^{2\gamma t}m\dot{q} \tag{6}$$

and the Hamiltonian

$$H(p,q,t) = p\dot{q} - L(q,\dot{q},t) = e^{-2\gamma t}\frac{p^2}{2m} + e^{2\gamma t}V(q)\,. \tag{7}$$

(c) Specializing to the harmonic potential (with $m = \omega \equiv 1$),

$$H(p,q,t) = e^{-2\gamma t}\,\frac{p^2}{2} + e^{2\gamma t}\,\frac{q^2}{2}\,. \tag{8}$$

A canonical transformation is a change of variables $Q = Q(q,p,t)$, $P = P(q,p,t)$ that preserve the form of the Hamilton equations. The change of variables must then be consistent with the variational principles

$$\delta\int_{t_1}^{t_2}\left(p\dot{q} - H(p,q,t)\right)dt = 0\,, \quad \delta\int_{t_1}^{t_2}\Big(P\dot{Q} - K(P,Q,t)\Big)\,dt = 0\,. \tag{9}$$

Compatibility of the two variational principles is ensured by

$$p\dot{q} - H(p,q,t) = P\dot{Q} - K(P,Q,t) + \frac{dF}{dt} \tag{10}$$

where F is an arbitrary function. Since we are looking for a K with no explicit time dependence, the compelling guess is

$$P = e^{-\gamma t} p\,, \quad Q = e^{\gamma t} q\,. \tag{11}$$

Substituting $p = e^{\gamma t} P$, $q = e^{-\gamma t} Q$ in (10),

$$P\dot{Q} - \gamma PQ - \frac{P^2}{2} - \frac{Q^2}{2} = P\dot{Q} - K(P,Q,t) + \frac{dF}{dt}\,, \tag{12}$$

we see that this indeed works with $F \equiv 0$ and

$$K(P,Q) = \frac{P^2}{2} + \frac{Q^2}{2} + \gamma PQ\,. \tag{13}$$

Another way to see that (11) is a legitimate canonical transformation is to check invariance of the symplectic form up to an exact term,

$$dP \wedge dQ = dp \wedge dq + d\left(\gamma(qdp + pdq)t\right)\,. \tag{14}$$

(d) The Hamiltonian equations for K are

$$\dot{P} = -\frac{\partial K}{\partial Q} = -Q - \gamma P\,, \quad \dot{Q} = \frac{\partial K}{\partial P} = P + \gamma Q\,, \tag{15}$$

from which we find the eom for Q,

$$\ddot{Q} + (1-\gamma^2)Q = 0\,. \tag{16}$$

The general solution in the underdamped case $\gamma < 1$ is

$$Q(t) = A\cos(\sqrt{1-\gamma^2}\,t) + B\sin(\sqrt{1-\gamma^2}\,t)\,, \tag{17}$$

with A and B integration constants. In terms of the original variable q,

$$q(t) = e^{-\gamma t}\left(A\cos(\sqrt{1-\gamma^2}\,t) + B\sin(\sqrt{1-\gamma^2}\,t)\right)\,, \tag{18}$$

which is indeed the familiar solution for an underdamped harmonic oscillator.

Electromagnetism 1

Rotating charged cylinder

An infinitely long and non-conducting cylindrical tube of radius R and surface charge density σ rotates around its symmetry axis with angular velocity ω.

(a) (3 points) Evaluate the electric and magnetic fields in and out of the cylindrical tube.

(b) (3 points) Now assume the cylinder to rotate from rest with an angular velocity $\omega(t) = \alpha t$. Evaluate the steady state electric and magnetic fields inside the cylindrical tube.

(c) (5 points) Characterize the energy flow around the cylinder.

(d) (9 points) Evaluate the work done by the fields How does your result compare to the change in the field energy per unit length inside the tube. Explain.

Solution

a. The electric and magnetic fields in and out are

$$\vec{E}_{\rm in} = \vec{0} \qquad \vec{E}_{\rm out} = \frac{4\pi\sigma\vec{r}}{R}$$
$$\vec{B}_{\rm out} = \vec{0} \qquad \vec{B}_{\rm in} = \frac{4\pi}{c}\sigma\omega R\,\hat{z} \tag{1}$$

b. Since the displacement current vanishes in and out (stationary electric fields see below), the magnetic fields in and out are

$$\vec{B}_{\rm out} = \vec{0} \qquad \vec{B}_{\rm in}(t) = \frac{4\pi}{c}\sigma\omega(t)R\,\hat{z} \tag{2}$$

The electric fields in and out follow from Lenz law $\vec{\nabla}\times\vec{E} = -\partial\vec{B}/c\partial t$ or

$$\oint \vec{E}\cdot d\vec{l} = -\frac{d\Phi}{cdt} \rightarrow \qquad \vec{E}_{\rm in} = -\frac{\dot{B}_{\rm in}r}{2c}\hat{\phi} \qquad \vec{E}_{\rm out} = -\frac{\dot{B}_{\rm in}R^2}{2cr}\hat{\phi} \tag{3}$$

c. The energy flow is captured by the Poynting vector in and out

$$\vec{S}_{\rm out} = \frac{c}{4\pi}\vec{E}_{\rm out}\times\vec{B}_{\rm out} = \vec{0}$$
$$\vec{S}_{\rm in} = \frac{c}{4\pi}\vec{E}_{\rm in}\times\vec{B}_{\rm in} = -\frac{c}{4\pi}E_{\rm in}B_{\rm in}\hat{r} = -\frac{B_{\rm in}\dot{B}_{\rm in}}{8\pi}\vec{r} = -\frac{B_{\rm in}^2}{8\pi}\frac{\dot{\omega}}{\omega}\vec{r} \tag{4}$$

so the energy flows from the fields outside-in. On the tube surface, the energy flux is

$$\frac{\mathcal{E}(t)}{A} = -\frac{B_{\rm in}^2(t)}{8\pi}\frac{R}{t} \tag{5}$$

d. The electric field exerts a torque on the surface of the cylindar. The torque τ per unit length H is

$$d\tau = ERdq = ER\sigma dA = -\left[\frac{\dot{B}_{\rm in}R}{2c}\right]R[\sigma dA = Rd\phi dH]$$
$$\frac{\tau}{H} = -\left[\frac{\dot{B}_{\rm in}R}{2c}\right][2\pi\sigma R^2] \tag{6}$$

The cumulative work W per length H done by the electric force at time t is

$$\frac{W(t)}{H} = \int_0^t \frac{\tau}{H}\,\omega(t)dt = -\frac{4\pi^2\sigma^2R^4}{c^2}\int_0^t \dot{\omega}\omega dt = -\frac{2\pi^2\sigma^2R^4\omega^2(t)}{c^2} \tag{7}$$

or the work per volume $V = \pi R^2 H$

$$\frac{W(t)}{V} = -\frac{2\pi(\sigma R\omega(t))^2}{c^2} = -\frac{B_{\text{in}}^2(t)}{8\pi} \tag{8}$$

which is minus the magnetic energy per unit volume being stored inside the tube. In storing this energy, outside work through spinning is needed.

Electromagnetism 2

Optically active media

An optically active medium can rotate the plane of polarization of light by allowing right and left circularly polarized waves that obey different dispersion relations. The electric susceptibility tensor at frequency ω of such a medium can be expressed as

$$\hat{\chi} = \begin{pmatrix} \chi_{11} & i\chi_{12} & 0 \\ -i\chi_{12} & \chi_{11} & 0 \\ 0 & 0 & \chi_{33} \end{pmatrix}$$

where $\hat{\chi}$ is related to the electric polarization $\boldsymbol{P}$ in the usual way: $P_i = \epsilon_0 \chi_{ij} E_j$. Here and below the fields are harmonic in time, $\boldsymbol{E}(t, \boldsymbol{r}) = \mathrm{Re}[e^{-i\omega t}\boldsymbol{E}(\boldsymbol{r})]$.

(a) (4 points) Derive the wave equation (analogous to the Helmholtz equation) satisfied by the electric field $\boldsymbol{E}(\boldsymbol{r})$ in this medium.

(b) (4 points) Now assume that a plane wave propagates in the medium in the z direction (which is also the 3-direction). Show that the propagating electromagnetic wave is transverse.

(c) (3 points) Show that the medium admits electromagnetic waves of two distinct wave vectors of magnitude k_R and k_L. Find these wave vectors in terms of ω and the necessary elements of $\hat{\chi}$.

(d) (4 points) Explicitly show that k_R and k_L correspond to the propagation of right- and left-circularly polarized electromagnetic waves.

(e) (5 points) Compute the ratio between the (time averaged) Poynting flux and the (time averaged) energy density for the right and left circularly polarized waves.

Solution

(a) Then we have the Maxwell equations

$$\begin{aligned}
\nabla \cdot \boldsymbol{E} + \nabla \cdot \boldsymbol{P} =& 0\,, && (1)\\
\nabla \times \boldsymbol{H} - \frac{1}{c}(-i\omega)\boldsymbol{D} =& 0\,, && (2)\\
\nabla \cdot \boldsymbol{B} =& 0\,, && (3)\\
\nabla \times \boldsymbol{E} + \frac{1}{c}(-i\omega)\boldsymbol{B} =& 0\,. && (4)
\end{aligned}$$

We take the curl of the last equation, use

$$\nabla \times \nabla \times \boldsymbol{E} = \nabla(\nabla \cdot \boldsymbol{E}) - \nabla^2 \boldsymbol{E}\,, \tag{5}$$

and exploit the second equation to find

$$-\nabla^2 \boldsymbol{E} + \nabla(\nabla \cdot \boldsymbol{E}) - \frac{\omega^2}{c^2}\boldsymbol{D} = 0\,. \tag{6}$$

Then we use, $\boldsymbol{D} = \boldsymbol{E} + \boldsymbol{P}$, a bit more explicitly to find finally a wave equation for E

$$-\partial_i \partial^i E_j + \partial_j(\partial_i E_i) - \frac{\omega^2}{c^2}\left(\delta_{jk} + \chi_{jk}\right) E_k\,. \tag{7}$$

(b) For a plane wave $E_j(x) = E_j e^{ikx}$ we have

$$k^2 E_j - k_j(k_i E_i) - \frac{\omega^2}{c^2}\left(\delta_{jk} + \chi_{jk}\right) E_k = 0\,. \tag{8}$$

Taking k in the z direction, and the indices a, b in the transverse direction, we find the equations of motion

$$\begin{aligned}
k_z^2 E_z - k_z^2 E_z + \frac{\omega^2}{c^2}\left(\delta_{zz} + \chi_{zz}\right) E_z =& 0\,, && (9)\\
k^2 E_a - \frac{\omega^2}{c^2}\left(\delta_{ab} + \chi_{ab}\right) E_b =& 0\,. && (10)
\end{aligned}$$

The first equation gives that $E_z = 0$, i.e. the waves are transverse.

(c) The second equation gives and eigen-equation for E_a. The non-trivial solutions are found when

$$\det \begin{pmatrix} k^2 - (\omega/c)^2 \chi_{11} & -i(\omega/c)^2 \chi_{12} \\ i(\omega/c)^2 \chi_{12} & k^2 - (\omega/c)^2 \chi_{11} \end{pmatrix} = 0\,, \tag{11}$$

which determines k for the specified frequency. This determines the dispersion curve[1]. We are solving an eigen system of the form

$$k^2 \begin{pmatrix} \vec{E}_1 \\ \vec{E}_2 \end{pmatrix} = \frac{\omega^2}{c^2} \begin{pmatrix} 1+\chi_{11} & i\chi_{12} \\ -i\chi_{12} & 1+\chi_{11} \end{pmatrix} \begin{pmatrix} \vec{E}_1 \\ \vec{E}_2 \end{pmatrix} . \tag{12}$$

The matrix we are finding the eigen-values of is of the form $(1+\chi_{11})\,\mathbb{I} - \chi_{12}\sigma_y$, where $\mathbb{I}$ is the identity matrix and σ_y is a Pauli-matrix. The eigenvalues of such a matrix are

$$k_{\mp}^2 = \frac{\omega^2}{c^2}\lambda_{\mp} , \qquad \lambda_{\mp} \equiv \left[(1+\chi_{11}) \mp \chi_{12}\right] . \tag{13}$$

(d) The corresponding eigenvectors are

$$\vec{E}_{\mp} = \begin{pmatrix} 1 \\ \pm i \end{pmatrix} . \tag{14}$$

So the minus solution is

$$E_a = \mathrm{Re}\left[\mathcal{A}\, e^{ik_{-}x - i\omega t} \begin{pmatrix} 1 \\ i \end{pmatrix} \right] , \tag{15}$$

where $\mathcal{A}$ is a complex amplitude. At $x=0$ (and setting the phase of $\mathcal{A}$ to zero) we have

$$E_x = \mathcal{A}\cos(\omega t) , \tag{16}$$

$$E_y = \mathcal{A}\sin(\omega t), \tag{17}$$

and thus the minus solution is right handed, k_R. The plus solution is left-handed k_L.

(e) Consider the *plus* plane wave (i.e. k_L)

$$\boldsymbol{E}(t,z) = \vec{E} e^{ikz - i\omega t} , \tag{18}$$

$$\boldsymbol{D}(t,z) = \vec{D} e^{ikz - i\omega t} , \tag{19}$$

$$\boldsymbol{B}(t,z) = \vec{B} e^{ikz - i\omega t} , \tag{20}$$

where it is understood that we are to take the real part, and we have set the amplitude to unity for simplicity.

Then since $\boldsymbol{D} = \boldsymbol{E} + \boldsymbol{P}$

$$\vec{D}_a = (\delta_{ab} + \chi_{ab})\vec{E}_b = \lambda_{+}\vec{E}_a . \tag{21}$$

Similarly, from $\nabla \times \boldsymbol{B} = -i\omega/c\, \boldsymbol{D}$

$$ik\, \hat{\boldsymbol{z}} \times \vec{B}^{+} = \frac{-i\omega}{c}\lambda_{+}\vec{E} \tag{22}$$

[1] Normally the dispersion curve is taken as $\omega(k)$. But both forms $\omega(k)$ or $k(\omega)$ are useful in different contexts.

So since

$$k_+^2 = \frac{\omega^2}{c^2}\lambda_+ \,, \tag{23}$$

We find

$$\hat{\boldsymbol{z}} \times \vec{B} = -\sqrt{\lambda_+}\vec{E} \,, \qquad \text{or} \qquad \vec{B} = \sqrt{\lambda_+}\,\hat{\boldsymbol{z}} \times \vec{E} \,. \tag{24}$$

Putting the ingredients together using

$$u = \frac{1}{2}(\boldsymbol{E}\cdot\boldsymbol{D} + \boldsymbol{B}\cdot\boldsymbol{H}) \qquad \boldsymbol{S} = c\boldsymbol{E}\times\boldsymbol{H} \,, \tag{25}$$

we find since $\boldsymbol{B} = \boldsymbol{H}$

$$\langle u \rangle = \frac{1}{4}\mathrm{Re}\left[\vec{E}\cdot\vec{D}^* + \vec{B}\cdot\vec{B}^*\right] \,, \tag{26}$$

$$= \frac{1}{2}\lambda_+\vec{E}\cdot\vec{E}^* \,. \tag{27}$$

The Poyting vector is

$$\langle \boldsymbol{S} \rangle = \frac{c}{2}\mathrm{Re}\left[\vec{E}\times\vec{B}^*\right] \,, \tag{28}$$

$$= \frac{1}{2}\sqrt{\lambda_+}\vec{E}\times(\hat{\boldsymbol{z}}\times\vec{E}) \,, \tag{29}$$

$$= \frac{1}{2}\sqrt{\lambda_+}\vec{E}\cdot\vec{E}^*\;\hat{\boldsymbol{z}} \,. \tag{30}$$

Then we find

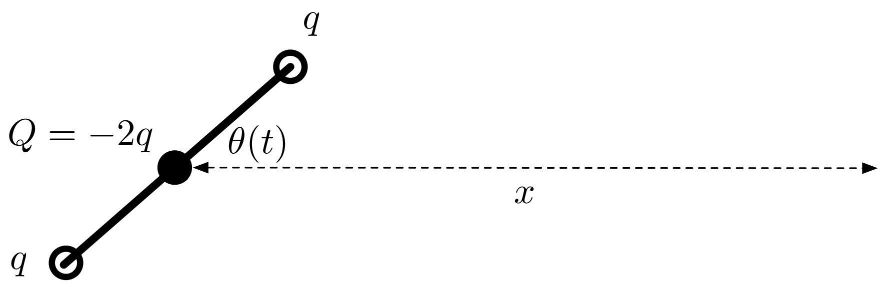
$$\langle \boldsymbol{S} \rangle = \frac{c}{\sqrt{\lambda_+}}\,u\;\hat{\boldsymbol{z}} \tag{31}$$

and recognize that there a similar expression for the minus mode, with the replacement $\lambda_+ \to \lambda_-$.

Electromagnetism 3

Rotations of a neutral rod

Two particles of charge q sit on the opposite sides of a rod of length $2a$, and a balancing charge $Q = -2q$ sits in the rod's center (see below). The orientation angle of the rod oscillates in time $\theta(t) = \pi\cos(\omega t)$, and the motion is non-relativistic. Here you will determine the electric and magnetic field at a distance x along the x axis.



For charge and current distributions, $\rho(\boldsymbol{r})$ and $\boldsymbol{J}(\boldsymbol{r})$, the multipole moments are:[2]

Multipole	Definition
Electric Monopole	$\int d^3\boldsymbol{r}\,\rho(\boldsymbol{r})$
Electric Dipole	$\int d^3\boldsymbol{r}\,\boldsymbol{r}\,\rho(\boldsymbol{r})$
Magnetic Dipole	$\frac{1}{2c}\int d^3\boldsymbol{r}\,(\boldsymbol{r}\times\boldsymbol{J}(\boldsymbol{r}))$
Electric Quadrupole	$\int d^3\boldsymbol{r}\,\rho(\boldsymbol{r})\,(3r^ir^j - \delta^{ij}r^2)$

We define $v_0 \equiv a\omega$ and $\lambda_0 \equiv c/\omega$, and note that $v_0/c = a/\lambda_0$.

(a) (3 points) Determine the listed multipole moments for the rod as a function of time.

(b) (3 points) For $a \ll x \ll \lambda_0$, determine the contribution to the electric and magnetic fields at position x from the magnetic dipole moment in terms of q, λ_0, a, x and fundamental constants.

(c) (6 points) For $a \ll x \ll \lambda_0$, *estimate* the magnitude of each listed multipole to the rod's electric and magnetic field in terms of q, λ_0, a, x and fundamental constants. Does your estimate confirm your calculation in (b)? Is the magnetic dipole the dominant contribution to the electric and magnetic field? Explain.

(d) (6 points) Determine the electric and magnetic field as a function of time for $x \gg \lambda_0$, and determine the instantaneous energy flux, S^x, at position x. Neglect the quadrupole moment.

(e) (2 points) Estimate the quadrupole contribution to the energy flux of (d). Was neglecting this contribution in part (d) justified? Explain.

[2]We are using Gaussian units. In SI units the magnetic moment does not have $1/c$.

Solution

(a) The electric monopole and dipole are zero. The magnetic dipole is in the z direction. The velocity of the charges at “left" end is $\boldsymbol{v}(t) = a\pi\omega \sin(\omega t)(-\hat{\boldsymbol{y}})$. The full dipole moment is a sum of the two moving contributions

$$\boldsymbol{m} = \sum \frac{q}{2c}\boldsymbol{r} \times \boldsymbol{v} = \frac{q}{c}\pi a^2\omega \sin(\omega t)(-\hat{\boldsymbol{z}})\,. \tag{32}$$

Thus

$$\boldsymbol{m} = \pi q \frac{a^2}{\lambda_0} \sin(\omega t)(-\hat{\boldsymbol{z}})\,. \tag{33}$$

To determine the quadrupole moment write the coordinate vectors of the two charges as:

$$\boldsymbol{r}_1 =(a\cos(\theta), a\sin\theta, 0), \tag{34}$$

$$\boldsymbol{r}_2 =-\,(a\cos(\theta), a\sin\theta, 0). \tag{35}$$

Then the contribution for $\boldsymbol{r}_1$ is

$$Q_{ij} = qa^2 \begin{pmatrix} 3\cos 2\theta - 1 & \cos\theta\sin\theta & 0 \\ \cos\theta\sin\theta & 3\sin^2\theta - 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \tag{36}$$

after a little bit of algebra and adding the contribution from the second charge we find

$$Q_{ij} = qa^2 \begin{pmatrix} \cos 2\theta & \sin 2\theta & 0 \\ \sin 2\theta & -\cos 2\theta & 0 \\ 0 & 0 & 0 \end{pmatrix} + qa^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}\,. \tag{37}$$

Only the first term depends on time, and ultimately contributes to the radiation. Note the period of the quadrupole moment is half that of the dipole moment, since Q_{ij} scales with 2θ instead of theta.

(b) The magnetic field from the dipole

$$\boldsymbol{B} = \frac{3(\boldsymbol{n}\cdot\boldsymbol{m})\boldsymbol{n} - \boldsymbol{m}}{4\pi r^3} \tag{38}$$

So in the current case $\boldsymbol{n} = \hat{\boldsymbol{x}}$ while $\boldsymbol{m} = m(t)(-\hat{\boldsymbol{z}})$ with $m(t) = q\pi a^2/\lambda_0 \sin(\omega t)$. So we have simply

$$\boldsymbol{B}(t, x) = \frac{q}{4}\frac{a^2}{x^3\lambda_0} \sin(\omega t)\hat{\boldsymbol{z}}\,. \tag{39}$$

The vector potential from a dipole is

$$\boldsymbol{A} = \frac{\boldsymbol{m} \times \hat{\boldsymbol{r}}}{4\pi r^2}\,, \tag{40}$$

and the induced electric field is

$$\boldsymbol{E} = -\frac{1}{c}\partial_t \boldsymbol{A}\,. \tag{41}$$

Then since $-(-\hat{\boldsymbol{z}}) \times \hat{\boldsymbol{x}} = \hat{\boldsymbol{y}}$ we have

$$\boldsymbol{E}(t, \boldsymbol{x}) = \frac{1}{c} \frac{\dot{m}}{4\pi r^2} \hat{\boldsymbol{y}} \,. \tag{42}$$

Putting in the value of $m(t)$ we have

$$\boldsymbol{E}(t, x) = \frac{q}{4} \frac{a^2}{x^2 \lambda_0^2} \cos(\omega t) \hat{\boldsymbol{y}} \tag{43}$$

(c) This is the near zone there. Clearly the electric monopole and dipole contributions are zero. The magnetic field from a magnetic dipole is

$$B \sim \frac{m}{x^3} \sim q \frac{a^2}{x^3 \lambda_0} \tag{44}$$

where $\lambda_0 \equiv \omega/c$. This is compatible with Eq. (39). The corresponding electric field is from Faraday's Law

$$\oint \boldsymbol{E} \cdot \mathrm{d}\ell = -\frac{1}{c} \partial_t \int \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{a} \,, \tag{45}$$

and thus we estimate

$$E \sim \frac{x}{c} \dot{B} \sim q \frac{a^2}{x^2 \lambda_0^2} \,, \tag{46}$$

which is consistent with Eq. (43).

The quadrupole field is

$$E \sim \frac{Q}{x^4} \sim q \frac{a^2}{x^4} \,. \tag{47}$$

The magnetic field comes from the Maxwell corrections to Ampere's Law

$$\oint \boldsymbol{B} \cdot \mathrm{d}\ell = \frac{1}{c} \partial_t \int \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{a} \,, \tag{48}$$

yielding

$$B \sim \frac{x}{c} \dot{E} \sim q \frac{a^2}{x^3 \lambda} \,. \tag{49}$$

Since $x \ll \lambda_0$, we see that the electric quadrupole's electric field is larger than that from the magnetic dipole. The quadrupole's magnetic field is comparable to the magnetic dipoles magnetic field, and thus can not be neglected.

(d) This is magnetic dipole radiation in the far field. We have

$$\boldsymbol{B} = \frac{1}{4\pi r c^2} \boldsymbol{n} \times \boldsymbol{n} \times \ddot{\boldsymbol{m}}(t - r/c) \,. \tag{50}$$

In the current case we have $\boldsymbol{m} = m(t)(-\hat{\boldsymbol{z}})$. So we find

$$\frac{1}{c^2} \ddot{m}(t) = -q\pi \frac{a^2}{\lambda_0^3} \,. \tag{51}$$

So

$$\boldsymbol{B}(t,x) = \frac{q}{4}\frac{a^2}{x\lambda_0^3}\sin(\omega(t - x/c))(-\hat{\boldsymbol{z}}) \tag{52}$$

We note that the magnetic field here in Eq. (52) is 180^o out of phase from the near field result in Eq. (39).

This is a radiation field. Thus the electric and magnetic field are equal in magnitude, in phase, and perpendicular to each other, as in a plane wave. Thus we have

$$\boldsymbol{E}(t,x) = \frac{q}{4}\frac{a^2}{x\lambda_0^3}\sin(\omega(t - x/c))(-\hat{\boldsymbol{y}}) \tag{53}$$

The Poynting flux is

$$S^x = (\boldsymbol{E}\times\boldsymbol{B})^x = \left(\frac{q}{4}\right)^2\left(\frac{a^2}{x\lambda_0^3}\right)^2\sin^2(\omega(t - x/c)\,. \tag{54}$$

(e) The power in quadrupole radiation follows as $\omega^6 \sim 1/\lambda_0^6$. This is the same order of magnitude as magnetic dipole radiation.

Quantum Mechanics 1

Coherent states of a harmonic oscillator

A one-dimensional quantum harmonic oscillator with coordinate x, mass m, and frequency ω is initially in its ground state $|0\rangle$. At time $t = 0$, a constant force F is abruptly applied to the oscillator.

(a) *(2 points)* Write down the Hamiltonian H of the oscillator at $t > 0$ (with the force term) through the creation/annihilation operators $a^\dagger, a$.

(b) *(4 points)* Find the new creation/annihilation operators $b^\dagger, b$ in terms of the old ones $a^\dagger, a$ such that the Hamiltonian H takes the form of the Hamiltonian of a free harmonic oscillator without an external force,

$$H = \hbar\omega b^\dagger b + \text{const}\,.$$

(c) *(3 points)* What is the state $|\psi(0)\rangle$ of the oscillator at time $t = 0$, right after the force F is applied? Write down the equation this state should satisfy in terms of the operator b.

(d) *(6 points)* Obtain an explicit expansion of the state $|\psi(0)\rangle$ in the basis $\{|n\rangle\}$ of the energy eigenstates of the b-oscillator and normalize it. From this expansion, write down the time-dependent normalized state $|\psi(t)\rangle$ of the oscillator at $t > 0$.

(e) *(4 points)* Using the expression for $|\psi(t)\rangle$, find the time dependence of the average coordinate $\langle x(t)\rangle$ and momentum $\langle p(t)\rangle$ of the oscillator.

(f) *(1 point)* Very briefly, interpret your result in part (e) from the point of view of the Ehrenfest theorem.

Solution

(a) Using the standard expression for the oscillator coordinate x in terms of the creation/annihilation operators $a^\dagger, a$, one can directly write the Hamiltonian as:

$$H = H_0 - Fx = \hbar\omega\left(a^\dagger a + \frac{1}{2}\right) - F\sqrt{\frac{\hbar}{2m\omega}}\left(a^\dagger + a\right) . \tag{1}$$

(b) From the expression for H above, one can see immediately that to include the force term in the main part of the Hamiltonian, one needs to introduce the annihilation operator b as

$$b = a - f\,, \quad f \equiv \frac{F}{(2\hbar m\omega^3)^{1/2}}.$$

(This corresponds to the shift of the potential minimum of the classical harmonic oscillator by the constant external force.) In terms of the operator b and Hermitian conjugate $b^\dagger$, the Hamiltonian H takes the form of the free harmonic oscillator:

$$H = \hbar\omega a^\dagger a - F\sqrt{\frac{\hbar}{2m\omega}}\left(a^\dagger + a\right) = \hbar\omega b^\dagger b + \text{const}\,. \tag{2}$$

(c) Since the force is switched on abruptly, the state $|\psi(0)\rangle$ of the oscillator right after switching on of the force F coincides with the initial state before the force was switched one, i.e., the ground state on the initial "a"-oscillator: $|\psi(0)\rangle = |0\rangle$. As usual, this state satisfies the condition $a|0\rangle = 0$, i.e.,

$$a|\psi(0)\rangle = 0\,.$$

In terms of the new annihilation operators b, this equation is:

$$b|\psi(0)\rangle = -f|\psi(0)\rangle\,,$$

i.e., the state $|\psi(0)\rangle$ represents a coherent state.

(d) To expand the state $|\psi(0)\rangle$ in the basis $\{|n\rangle\}$ of the energy eigenstates of the "b"-oscillator:

$$|\psi(0)\rangle = \sum_n c_n|n\rangle$$

we note that the states in this basis are given by the standard relations for the energy eigenstates of a harmonic oscillator:

$$|n\rangle = \frac{1}{\sqrt{n!}}(b^\dagger)^n|0\rangle\,,$$

where, as usual, the ground state of the oscillator is defined by the equation

$$b|0\rangle = 0\,.$$

The expansion coefficients c_n are found directly:

$$c_n = \langle n|\psi(0)\rangle = \frac{1}{\sqrt{n!}}\langle 0|b^n|\psi(0)\rangle = \frac{(-f)^n}{\sqrt{n!}}\langle 0|\psi(0)\rangle\,,$$

and after proper normalization

$$c_n = \frac{(-f)^n}{\sqrt{n!}}e^{-f^2/2}.$$

Finally, taking into account the standard time evolution of the energy eigenstates, we obtain the wavefunction of the oscillator in the basis of the new energy eigenstates:

$$|\psi(t)\rangle = \sum_n \frac{[-fe^{-i\omega t}]^n}{\sqrt{n!}}|n\rangle e^{-f^2/2}. \tag{3}$$

(e) Using the wavefunction $|\psi(t)\rangle$ and the standard properties of the creation/annihilation operators one finds:

$$\langle\psi(t)|b|\psi(t)\rangle = -fe^{-i\omega t}, \quad \langle\psi(t)|b^\dagger|\psi(t)\rangle = -fe^{i\omega t},$$

and from this, the average coordinate $\langle x\rangle(t)$ and momentum $\langle p\rangle(t)$:

$$\langle x(t)\rangle = \sqrt{\frac{\hbar}{2m\omega}}(2f - 2f\cos\omega t) = \frac{F}{m\omega^2}(1-\cos\omega t)\,, \tag{4}$$

$$\langle p(t)\rangle = -i\sqrt{\frac{\hbar m\omega}{2}}f(e^{i\omega t} - e^{-i\omega t}) = \frac{F}{\omega}\sin\omega t\,. \tag{5}$$

(f) Equations obtained in (e) coincide precisely with those describing the evolution of coordinate and momentum of a classical harmonic oscillator, in agreement with the Ehrenfest theorem.

Quantum Mechanics 2

Jordan-Wigner transformation

The Jordan-Wigner transformation is a mapping between spin 1/2 operators and fermion creation/annihilation operators. Qualitatively, for a system of spins on sites in a lattice, the Jordan-Winger transformation maps spin up/spin down states on each site to the filled/empty fermion occupation states on the same site. (All spin operators in this problem are defined in units of $\hbar$.)

1. Quantitatively, we consider first a single site with spin 1/2. Take the mapping $|\uparrow\rangle \equiv f^\dagger|0\rangle$ and $|\downarrow\rangle \equiv |0\rangle$, where $f^\dagger$ and its Hermitian conjugate f are the raising and lowering operators for the fermion on this site. The f operators obey the standard fermionic commutation relations, and state $|0\rangle$ is the "empty" state: $f|0\rangle = 0$.

(a) (3 points) From the connection between the spin states and f operators given above, show that we can write $S_z = f^\dagger f - \frac{1}{2}$. What are the expressions for S_x and S_y in terms of the f operators?

(b) (3 points) Show that the spin commutation relations are respected by transformation to the fermion operators. (It is sufficient to demonstrate explicitly only one relation.) What is the anticommutator $\{S^+, S^-\}$?

2. Now consider a chain of spin 1/2's in one dimension on a lattice of sites $j = 1...N$, each separated by distance a.

(a) (1 point) State in one sentence what is wrong with the mapping between spins and fermions in part (1) in this case.

Hint: Compare the commutation relation between spins and between fermions *on different sites* j and l.

(b) (5 points) To overcome this problem, one generalizes expressions from part (1), and defines the spin raising and lowering operators at site j through the fermion operators as

$$S_j^+ = f_j^\dagger e^{i\phi_j}, \quad S_j^- = f_j e^{-i\phi_j},$$

where the phase operator contains the sum over all fermion occupancies n at sites to the left of j: $\phi_j = \pi \sum_{l<j} n_l$. Show that the "string operator" $e^{i\phi_j}$ solves the issue in 2(a).

Hint: First show that f_j and $f_j^\dagger$ anticommute with $e^{i\pi n_j}$. What about $e^{i\pi n_j}$ and f_l for $l \neq j$? Finally, show that the spin raising and lowering operators as defined above commute on different sites, when expressed through f operators. (As in 1b, one commutation relation would be sufficient.)

3. Consider the one-dimensional, anisotropic Heisenberg model with two interaction constants J and J_z:

$$H = -\frac{J}{2}\sum_j [S_{j+1}^+ S_j^- + \text{H.c.}] - J_z \sum_j S_j^z S_{j+1}^z .$$

(a) (1 point) Show that in terms of the fermion operators, the quadratic part of this Hamiltonian (the one that does not contain "interactions," i.e., expressions with more than two fermion operators) is:

$$H = -\frac{J}{2}\sum_j (f_{j+1}^\dagger f_j + \text{H.c.}) + J_z \sum_j f_j^\dagger f_j \, .$$

(b) (2 points) Next, the noninteracting Hamiltonian from 3(a) can be Fourier transformed to the momentum space via $f_j = \frac{1}{\sqrt{N}}\sum_k s_k e^{ikR_j}$, where $R_j = ja$. Show that the result of this transformation is

$$H = \sum_k \omega_k s_k^\dagger s_k \, ,$$

and find the energy ω_k as a function of J_z, J, k, and a.

(c) (5 points) ω_k in part 3(b) defines the "magnon excitation energy." Take $J_z = J > 0$. Describe very briefly the ground state of the model. Sketch the magnon dispersion (i.e., ω_k versus k) between $k = -\pi/a$ and $k = \pi/a$. What does the value of ω_k at $k = 0$ say about the direction of spins in the ground state?

Solution

(1) (a) For a spin 1/2, the operator S_z is diagonal in the basis $\{|\uparrow\rangle, |\downarrow\rangle\}$ and has eigenvalues $\pm 1/2$, i.e.:

$$S_z|\uparrow\rangle = \frac{1}{2}|\uparrow\rangle, \quad S_z|\downarrow\rangle = -\frac{1}{2}|\downarrow\rangle \, .$$

From the given analogy to the fermion states, and the standard properties of the fermion operators, we see that the operator with the same properties (diagonal, with eigenvalues $\pm 1/2$ is indeed $f^\dagger f - \frac{1}{2}$:

$$[f^\dagger f - \frac{1}{2}]|0\rangle = -\frac{1}{2}|0\rangle \, , \quad [f^\dagger f - \frac{1}{2}]f^\dagger|0\rangle = \frac{1}{2}f^\dagger|0\rangle \, .$$

Next, the properties of the fermion operators imply that

$$f(f^\dagger|0\rangle) = ff^\dagger|0\rangle = (1 + f^\dagger f)|0\rangle = |0\rangle \, .$$

Combined with the definition of $f^\dagger$ above, we see that in terms of the spin operators: $f^\dagger = S^+$ and $f = S^-$. The properties of the spin operators then mean that,

$$S_x = \frac{1}{2}(f^\dagger + f) \, , \quad S_y = \frac{1}{2i}(f^\dagger - f) \, .$$

(b) The spin operators should have the relations $[S_a, S_b] = i\epsilon_{abc}S_c$. For example:

$$[S_x, S_y] = \frac{i}{4}[f^\dagger + f, f - f^\dagger] = \frac{i}{4}(f^\dagger f - ff^\dagger - ff^\dagger + f^\dagger f) = \frac{i}{2}(2f^\dagger f - 1) = iS_z \, ,$$

and so on. The anticommutator $\{S^+, S^-\}$ is directly

$$\{S^+, S^-\} = \{f^\dagger, f\} = 1 \, ,$$

and, as should be, coincides with the value it has for spin 1/2:

$$\{S^+, S^-\} = 2(S^2 - S_z^2) = 2(\frac{3}{4} - \frac{1}{4}) = 1 \, .$$

(2) (a) Spin operators for different sites, as operators of the independent systems, commute, while fermion operators anticommute even on different sites.

(b) Consider a state $|\psi\rangle$ in which the site j is filled by a fermion, i.e. $|\psi\rangle = |...1_j...\rangle$. Then $fe^{i\pi n_j}|\psi\rangle = -f|\psi\rangle$, while

$$e^{i\pi n_j}f|\psi\rangle = e^{i\pi n_j}|...0_j...\rangle = |...0_j...\rangle = f|\psi\rangle \, .$$

Thus, $\{e^{i\pi n_j}, f_j\} = f_j - f_j = 0$. An analogous argument holds for $f^\dagger$. It is clear that the phase factor at any other site $l \neq j$ commutes with f_j and $f_j^\dagger$. Thus we see that $\{e^{i\phi_j}, f_l\} = 0$ if $l < j$, and $[e^{i\phi_j}, f_l] = 0$ if $l \geq j$.

From this, taking $j < k$, we get:

$$\begin{aligned} [S_j^+, S_k^+] &= [f_j^\dagger e^{i\phi_j}, f_k^\dagger e^{i\phi_k}] \\ &= f_j^\dagger [e^{i\phi_j}, f_k^\dagger e^{i\phi_k}] + [f_j^\dagger, f_k^\dagger e^{i\phi_k}] e^{i\phi_j} \\ &= [f_j^\dagger, f_k^\dagger e^{i\phi_k}] e^{i\phi_j} = 0 \,, \end{aligned} \tag{1}$$

where we used the fact that $e^{i\phi_j}$ commutes with fermions on site j and k, and that $f_j^\dagger$ anticommutes with both $f_k^\dagger$ and $e^{i\phi_k}$, and therefore, commutes with their product. A similar argument holds for $[S_j^-, S_k^-]$ and $[S_j^+, S_k^-]$.

(3) (a) For the first term in the Hamiltonian, we have

$$-\frac{J}{2}\sum_j S_{j+1}^+ S_j^- = -\frac{J}{2}\sum_j f_{j+1}^\dagger e^{i\pi n_j} f_j = -\frac{J}{2}\sum_j f_{j+1}^\dagger f_j \,.$$

The second term becomes

$$-J_z \sum_j (f_{j+1}^\dagger f_{j+1} - \frac{1}{2})(f_j^\dagger f_j - \frac{1}{2}) \,.$$

Neglecting terms quartic in fermion operators, we get $J_z \sum_j f_j^\dagger f_j$ for this expression, and with this, the sought expression for the quadratic part of the Hamiltonian.

(b) The first term in the Hamiltonian is 3(a) becomes

$$-\frac{J}{2N}\sum_k (e^{ika} + e^{-ika}) s_k^\dagger s_k = -\frac{J}{N}\sum_k \cos(ka) s_k^\dagger s_k \,.$$

The second term is simply $J_z \sum_k s_k^\dagger s_k$. Thus, $H = \sum_k \omega_k s_k^\dagger s_k$, where

$$\omega_k = \frac{1}{N}[J_z - J\cos(ka)] \,.$$

(c) In the case $J = J_z > 0$, this gives $\omega_k > 0$, for $k \neq 0$. Therefore, the energy is increased for any non-uniform spin configuration. Thus, the ground state of the system is all spins pointing in the same direction. The magnon dispersion is sketched in Fig. 1. The fact that $\omega_{k=0} = 0$ means that there is no energy penalty for changing the direction of all spins simultaneously, i.e., the spins in the ground state should point in the same but arbitrary direction. This makes physical sense, since we have removed any anisotropy by setting $J = J_z$.

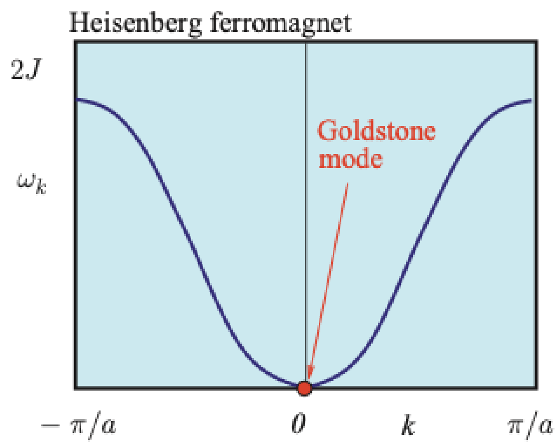


Figure 1: Magnon dispersion for part 3(c).

Quantum Mechanics 3

BEC superfluiditiy

The ground state of a quantum-degenerate gas (Bose-Einstein condensate) of bosonic atoms of mass m in an external potential V is described by a macroscopic wavefunction $\Psi(\vec{r},t) = \psi(\vec{r})e^{-i\mu t/\hbar}$ featuring a chemical potential μ due to a nonlinear effective interaction $g|\psi^2|$ (with $g > 0$) between the atoms. The zero-point energy in the potential is negligible.

First, consider an effectively 1D scenario and a box-shaped potential $V(z)$ to derive some basic properties of the condensate. One of the walls of the box is at $z = 0$, with $V = \infty$ for $z < 0$ and $V = 0$ for $z \geq 0$.

1. (a) (2 points) Write down the Schrödinger equation for $\psi(z)$. Find the stationary bulk density $n_0 = |\psi_0|^2 = |\psi(\infty)|^2$ far away from the wall where edge effects can be neglected.

(b) (3 points) Qualitatively sketch $n(z) = |\psi(z)|^2$ near the wall, and discuss the role of kinetic and interaction energy terms. Using the ansatz $n(z) = n_0 \tanh^2(x/[\sqrt{2}\xi])$, determine the healing length ξ.

2. Far from the walls, the condensate supports perturbative excitations

$$\delta\psi = e^{-i\mu t/\hbar}[ue^{i(kz-\omega t)} - v^* e^{-i(kz-\omega t)}]\,,$$

so that the total wavefunction is given by $\psi(z,t) = [\psi_0 + \delta\psi(z,t)]e^{-i\mu t/\hbar}$, with $|\delta\psi| \ll |\psi_0|$.

(a) (3 points) Derive a linearized Schrödinger equation for $\delta\psi$, keeping terms to first order in $\delta\psi$.

(b) (4 points) Show that the dispersion relation $\omega(k)$ of the perturbations is given by $\omega(k) = \left[k^2/2m\,(\hbar^2 k^2/2m + 2\mu)\right]^{1/2}$.

(c) (3 points) Using the result for $\omega(k)$, discuss the character of the excitations for small and large momenta (i.e. $k \ll 1/\xi$ and $k \gg 1/\xi$).

The result for $\omega(k)$ is valid also in higher dimensions. Now consider an impurity atom moving through a 3D condensate with velocity $\vec{v}$.

3. (a) (2 points) Calculate the Doppler shift of the frequency $\omega(k)$ of an excitation propagating in $\vec{k}/k$ direction in the atom's reference frame (you may assume $k \ll 1/\xi$).

(b) (3 points) The Doppler shift allows for resonant coupling between the impurity motion and the excitation, once v exceeds a critical velocity v_c. Calculate v_c for strong interactions, and in the absence of interactions. What is the friction force for motion below v_c? Justify your answer.

Solution

1. Start with the Schrödinger equation $i\hbar\partial_t\psi = [-\hbar^2\partial_z^2/2m + g|\psi(z)|^2]\psi(z)$ with $i\hbar\psi = \mu\psi(z)$. Far away from the wall, the wavefunction is flat such that $\mu\psi = g|\psi|^2\psi$ and hence $n_0 = |\psi_0|^2 = \mu/g$. The wavefunction has to vanish near the wall, reaching zero. While this reduces the interaction energy near the wall, it increases kinetic energy due to bending of the wavefunction, giving rise to the characteristic length scale ξ. Plugging the ansatz $\psi = \sqrt{n_0}\tanh(x/[\sqrt{2}\xi])$ into the Schrödinger equation yields

$$\xi = \hbar/\sqrt{2m\mu}. \tag{1}$$

2. Substituting the wavefunction $\psi(z,t) = [\psi_0 + \delta\psi(z,t)]e^{-i\mu t/\hbar}$ into the time-dependent Schrödinger equation and only keeping terms $\propto \delta\psi$ yields

$$i\hbar\delta\dot{\psi} = -\frac{\hbar^2\partial_z^2\delta\psi}{2m} + 2g|\psi_0|^2\delta\psi + g\psi_0^2\delta\psi^* \tag{2}$$

After inserting the expression for $\delta\psi(z,t)$, group all terms $\propto e^{-i\omega t}$ and $\propto e^{i\omega t}$ and require them to fulfil the equation separately for all times. This yields

$$0 = \left(\frac{\hbar^2k^2}{2m} + \mu - \hbar\omega\right)u - \mu v \tag{3}$$

$$0 = -\mu u + \left(\frac{\hbar^2k^2}{2m} + \mu + \hbar\omega\right)v \tag{4}$$

where we have already taken the complex conjugate of the last equation (which contained v^* and u^*). This system only has a solution for u and v if the determinant is zero; this immediately gives the desired equation for $\omega(k)$ (known as the Bologliubov dispersion relation). Note that the chemical potential is related to the healing length as $2\mu = (\hbar/\xi)^2/2m$. For small momentum, $k \ll 1/\xi$, the dispersion relation thus is linear (sound-like)

$$\omega(k) \approx \sqrt{\frac{\mu}{m}}\,k \tag{5}$$

whereas for large momentum, $k \gg 1/\xi$, it approximates that of a single free atom,

$$\omega(k) \approx \frac{\hbar k^2}{2m} \tag{6}$$

One can thus see that collective excitations of the condensate are only possible if the bending of the wavefunction is soft on the scale of the healing length.

3. In the reference frame of the moving impurity atom, the perceived frequency of an excitation propagating in the direction of impurity motion is $\omega'(k) = \omega(k)(1 - v/c_s)$ where $c_s = \sqrt{\mu/m}$ is the speed of sound. Thus for wave-like excitations with $\omega(k) = c_s k$ we have

$\omega'(k) = \omega(k) - kv$, which generalizes to $\omega'(k) = \omega(k) - \vec{k} \cdot \vec{v}$ in non-collinear situations. The condition for resonant excitation, $\omega'(k) = 0$ can thus be fulfilled (Landau criterion) for

$$v \geq v_c = \frac{\omega(k)}{k}, \tag{7}$$

which for $k \ll 1/\xi$ coincides with the speed of sound, $\sqrt{\mu/m}$. Above v_c, the motion can resonantly produce Bogoliubov excitations (given off sideways), which leads to damping of the motion. Below v_c, no excitations can be produced and the motion is frictionless (superfluidity). For vanishing interactions, v_s goes to zero with the chemical potential. In this case, excitations (in the form of single recoiling atoms produced by collisions with the impurity) can now occur for any non-zero velocity, and superfluidity goes away.

Statistical Mechanics 1

Phonons

Consider a cubic array of atoms in d dimensions, where each atom is labelled by its equilibrium position, $\mathbf{R} = \sum_{i=1}^{d} a n_i \mathbf{e}_i$, where a is the lattice spacing, n_i is an integer, and $\mathbf{e}_i$ is a unit vector in the direction indicated by i. As the atoms vibrate about their equilibrium positions, let $\mathbf{x_R}$ be the displacement of the atom whose equilibrium position is $\mathbf{R}$ from equilibrium, i.e., $\mathbf{x_R} = \mathbf{0}$ indicates the atom is sitting at its equilibrium position. Let $\mathbf{p_R}$ be the momentum conjugate to $\mathbf{x_R}$. The simplest description of the energy of the atomic vibrations is by imagining the atoms are connected by springs of spring constant ω. Then the Hamiltonian describing the energy of the crystal as the atoms vibrate is a sum of their kinetic and potential energies:

$$H = \sum_{\mathbf{R}} \frac{\mathbf{p}_{\mathbf{R}}^2}{2m} + \sum_{\mathbf{R}} \sum_{i=1}^{d} \frac{m\omega^2}{2} \left(\mathbf{x_R} - \mathbf{x}_{\mathbf{R}+a\mathbf{e}_i}\right)^2, \tag{1}$$

where m is the mass of an atom and the sum is over all equilibrium lattice positions $\mathbf{R}$.

Assume periodic boundary conditions in all directions, so that $\mathbf{R} \equiv \mathbf{R} + Na\mathbf{e}_i$. Then the Hamiltonian is translationally invariant and can be diagonalized using the Fourier transforms:

$$\begin{aligned} \mathbf{x_R} &= \frac{1}{\sqrt{N^d}} \sum_{\mathbf{q}} e^{i\mathbf{q}\cdot\mathbf{R}} \vec{\phi}_{\mathbf{q}} \\ \mathbf{p_R} &= \frac{1}{\sqrt{N^d}} \sum_{\mathbf{q}} e^{i\mathbf{q}\cdot\mathbf{R}} \vec{\pi}_{\mathbf{q}}. \end{aligned} \tag{2}$$

Notice that $\vec{\phi}$ and $\vec{\pi}$ are d-component vectors because $\mathbf{x}$ and $\mathbf{p}$ are d-component vectors. The periodic boundary conditions require $\mathbf{q} = \frac{2\pi}{Na} \sum_{i=1}^{d} m_i \mathbf{e}_i$, where m_i is an integer defined mod N.

1. Using the Fourier transforms in Eq. (2),

(a) (2 points) Show that the Hamiltonian in Eq. (1) can be written as a sum of decoupled harmonic oscillators for each $\mathbf{q}$:

$$H = \sum_{\mathbf{q}} \left[\frac{\vec{\pi}_{\mathbf{q}} \cdot \vec{\pi}_{-\mathbf{q}}}{2m} + \frac{m\omega_{\mathbf{q}}^2}{2} \vec{\phi}_{\mathbf{q}} \cdot \vec{\phi}_{-\mathbf{q}} \right] \tag{3}$$

(b) (2 points) Express $\omega_{\mathbf{q}}$ in terms of ω, $\mathbf{q}$, and a.

2. The mean squared atomic displacements are given by: $x_{rms}^2 \equiv \langle \mathbf{x_R} \cdot \mathbf{x_R} \rangle$.

(a) (3 points) Write an expression for x_{rms}^2 at finite temperature as a sum over $\mathbf{q}$.

(b) (2 points) Show that in the high temperature limit where $k_B T \gg \hbar\omega_{\mathbf{q}}$,

$$x_{rms}^2 = \frac{d}{N^d} \sum_{\mathbf{q}} \frac{k_B T}{m\omega_{\mathbf{q}}^2} \tag{4}$$

(c) (3 points) In the Debye approximation, the phonon frequency is approximated by $\omega_{\mathbf{q}} \to \omega a|\mathbf{q}|$. Compute x_{rms}^2 in three dimensions in the Debye approximation by converting the sum to an integral via $\sum_{\mathbf{q}} \cdots \to N^d a^d \int \frac{d^d q}{(2\pi)^d} \cdots$ and taking π/a as a large-momentum cut-off for q.

Hint: it may be helpful to define the raising and lowering operators:

$$\begin{aligned} a_{\mathbf{q},i} &= \sqrt{\frac{m\omega_{\mathbf{q}}}{2\hbar}} \left(\phi_{\mathbf{q},i} + \frac{i}{m\omega_{\mathbf{q}}} \pi_{\mathbf{q},i} \right) \\ a_{\mathbf{q},i}^{\dagger} &= \sqrt{\frac{m\omega_{\mathbf{q}}}{2\hbar}} \left(\phi_{-\mathbf{q},i} - \frac{i}{m\omega_{\mathbf{q}}} \pi_{-\mathbf{q},i} \right), \end{aligned} \tag{5}$$

where $\phi_{\mathbf{q},i}$ indicates the i^{th} component of $\vec{\phi}_{\mathbf{q}}$ and similarly for $\pi_{\mathbf{q},i}$. The raising and lowering operators satisfy $[a_{\mathbf{q},i}, a_{\mathbf{q}',j}^{\dagger}] = \delta_{\mathbf{q},\mathbf{q}'}\delta_{ij}$.

3. Lindemann's criterion predicts that a crystal will melt at the temperature where the atomic displacements reach the same scale as the lattice spacing; specifically, when $x_{rms}^2 = c_L a^2$, where c_L is a phenomenological constant.

(a) (3 points) Use Lindemann's criterion to compute the melting temperature of a three-dimensional crystal in the Debye approximation in terms of c_L, using the results from 2(c).

(b) (3 points) Repeat part 2(c) for two dimensions, using $\pi/(Na)$ as a small-momentum cut-off and π/a as a large-momentum cut-off. Show that x_{rms} diverges with system size.

(c) (2 points) What does part 3(b) imply for the stability of a two-dimensional crystal in the thermodynamic limit?

Solution

1. Plugging the Fourier transforms in Eq. (2) into Eq. (1) yields:

$$H = \sum_{\mathbf{R}} \frac{1}{N^d} \sum_{\mathbf{q},\mathbf{q}'} e^{i\mathbf{q}\cdot\mathbf{R}+i\mathbf{q}'\cdot\mathbf{R}} \frac{\vec{\pi}_{\mathbf{q}}\cdot\vec{\pi}_{\mathbf{q}'}}{2m}$$

$$+ \sum_{\mathbf{R}} \frac{1}{N^d} \sum_{\mathbf{q},\mathbf{q}'} \sum_{i=1}^{d} \frac{m\omega^2}{2} \times$$

$$\left(e^{i\mathbf{q}\cdot\mathbf{R}+i\mathbf{q}'\cdot\mathbf{R}} - e^{i\mathbf{q}\cdot\mathbf{R}-i\mathbf{q}'\cdot(\mathbf{R}+a\mathbf{e}_i)} - e^{i\mathbf{q}\cdot(\mathbf{R}+a\mathbf{e}_i)+i\mathbf{q}'\cdot\mathbf{R}} + e^{i\mathbf{q}\cdot(\mathbf{R}+a\mathbf{e}_i)+i\mathbf{q}'\cdot(\mathbf{R}+a\mathbf{e}_i)}\right) \vec{\phi}_{\mathbf{q}}\cdot\vec{\phi}_{\mathbf{q}'} \quad (6)$$

$$= \sum_{\mathbf{q}} \left[\frac{\vec{\pi}_{\mathbf{q}}\cdot\vec{\pi}_{-\mathbf{q}}}{2m} + \sum_{i=1}^{d} \frac{m\omega^2}{2} \left(2 - 2\cos(a\mathbf{q}\cdot\mathbf{e}_i)\right) \vec{\phi}_{\mathbf{q}}\cdot\vec{\phi}_{-\mathbf{q}} \right] \quad (7)$$

$$= \sum_{\mathbf{q}} \left[\frac{\vec{\pi}_{\mathbf{q}}\vec{\pi}_{-\mathbf{q}}}{2m} + \frac{m\omega_{\mathbf{q}}^2}{2} \vec{\phi}_{\mathbf{q}}\cdot\vec{\phi}_{-\mathbf{q}} \right], \quad (8)$$

where

$$\omega_{\mathbf{q}}^2 = 4\omega^2 \sum_{i=1}^{d} \sin^2 \frac{a\mathbf{q}\cdot\mathbf{e}_i}{2} \quad (9)$$

2. (a) By translation symmetry, $x_{rms}^2 = \langle \mathbf{x_R}\cdot\mathbf{x_R}\rangle = \langle \mathbf{x_0}\cdot\mathbf{x_0}\rangle$, which can be computed using the Fourier transform in Eq. (2) and the raising and lowering operators in Eq. (5):

$$x_{rms}^2 = \frac{1}{N^d} \sum_{\mathbf{q},\mathbf{q}'} \langle \vec{\phi}_{\mathbf{q}}\cdot\vec{\phi}_{\mathbf{q}'}\rangle \quad (10)$$

$$= \frac{1}{N^d} \sum_{\mathbf{q},\mathbf{q}'} \sum_{i=1}^{d} \frac{\hbar}{2m\sqrt{\omega_{\mathbf{q}}\omega_{\mathbf{q}'}}} \langle \left(a_{\mathbf{q},i} + a^\dagger_{-\mathbf{q},i}\right)\left(a_{\mathbf{q}',i} + a^\dagger_{-\mathbf{q}',i}\right)\rangle \quad (11)$$

Since the harmonic oscillators are decoupled, the right-hand-side will be zero unless $\mathbf{q}' = -\mathbf{q}$. Therefore:

$$x_{rms}^2 = \frac{1}{N^d} \sum_{\mathbf{q}} \sum_{i=1}^{d} \frac{\hbar}{2m\omega_{\mathbf{q}}} \langle 1 + n_{\mathbf{q},i} + n_{-\mathbf{q},i}\rangle \quad (12)$$

$$= \frac{d}{N^d} \sum_{\mathbf{q}} \frac{\hbar}{2m\omega_{\mathbf{q}}} \left(1 + \frac{2}{e^{\beta\hbar\omega_{\mathbf{q}}} - 1}\right), \quad (13)$$

where $n_{\mathbf{q},i} \equiv a^\dagger_{\mathbf{q},i} a_{\mathbf{q},i}$, $\beta \equiv 1/(k_B T)$ and we have used the fact that $\omega_{\mathbf{q}} = \omega_{-\mathbf{q}}$ and $\langle n_{\mathbf{q},i}\rangle = \frac{1}{e^{\beta\hbar\omega_{\mathbf{q}}}-1}$ following the Bose-Einstein distribution.

(b) In the high-temperature limit when $k_B T \gg \hbar\omega_{\mathbf{q}}$, we make the approximation $e^{\beta\hbar\omega_{\mathbf{q}}} - 1 \to \beta\hbar\omega_{\mathbf{q}}$ and Eq. (13) simplifies:

$$x_{rms}^2 \xrightarrow{k_B T \gg \hbar\omega_{\mathbf{q}}} \frac{d}{N^d} \sum_{\mathbf{q}} \frac{k_B T}{m\omega_{\mathbf{q}}^2} \quad (14)$$

(c) In the Debye approximation where $\omega_{\mathbf{q}} \to \omega a|\mathbf{q}|$:

$$x_{rms}^2 \to \frac{d}{N^d} \sum_{\mathbf{q}} \frac{k_B T}{m\omega^2 a^2 q^2} \tag{15}$$

In three dimensions, converting the sum to an integral yields:

$$\begin{aligned} x_{rms}^2 &\to 3a^3 \int \frac{d^3q}{(2\pi)^3} \frac{k_B T}{m\omega^2 a^2 q^2} && (16)\\ &= \frac{3a}{2\pi^2} \int_0^{\pi/a} dq \frac{k_B T}{m\omega^2} && (17)\\ &= \frac{3}{2\pi} \frac{k_B T}{m\omega^2} && (18) \end{aligned}$$

3. (a) Using Lindemann's criterion and x_{rms}^2 in three dimensions from Eq. (18), the melting temperature will occur when

$$\frac{3}{2\pi} \frac{k_B T_L}{m\omega^2} = c_L a^2 \Rightarrow T_L = \frac{2\pi m\omega^2 c_L a^2}{3k_B} \tag{19}$$

(b) Applying the expression for x_{rms}^2 in Eq. (15) to two dimensions yields:

$$\begin{aligned} x_{rms}^2 &\to 2a^2 \int \frac{d^2q}{(2\pi)^2} \frac{k_B T}{m\omega^2 a^2 q^2} && (20)\\ &= \frac{k_B T}{\pi m\omega^2} \int_{\frac{\pi}{Na}}^{\frac{\pi}{a}} \frac{dq}{q} && (21)\\ &= \frac{k_B T}{\pi m\omega^2} \ln(N), && (22) \end{aligned}$$

which diverges in the large N limit.

(c) Eq. (22) shows that fluctuations in atomic position diverge in the large N limit for a two-dimensional crystal. This means that strictly speaking, a two-dimensional crystal is unstable to fluctuations, i.e., it cannot exist in the thermodynamic limit. Applying Lindemann's criterion shows that the melting temperature occurs when $T_L \ln(N) \approx c_L a^2$, i.e., $T_L \to 0$ as $N \to \infty$ as a is held fixed. Thus, we reach the same conclusion that a crystal is unstable in two dimensions.

Statistical Mechanics 2

Mean field behavior of the spin-2 Ising model

Consider a spin-2 Ising model with the Hamiltonian:

$$\mathcal{H} = -\sum_{\langle i,j \rangle} J_{ij} S_i S_j - H \sum_i S_i \tag{1}$$

where spins $S_i \in$ {-2,-1,0,1,2} are on a simple cubic lattice and summation is performed over pairs of spins $\langle i,j \rangle$ (once each) that have only the nearest-neighbor coupling $J_1/k_B = 40$K and the next-nearest neighbor coupling $J_2/k_B = 10$K. Note that every spin has 6 nearest neighbors and 12 next-nearest neighbors.

(a) (5 points) Derive the mean field Hamiltonian by writing $S_i = m + \delta S_i$ (with $m = \langle S_i \rangle$ and $\delta S_i = S_i - m$) and then neglecting terms in quadratic fluctuations.

(b) (4 points) Find the mean field free energy $F(T, H, N, m)$

(c) (4 points) Find the mean field equation for m.

(d) (5 points) Find the mean field transition temperature T_c when $H = 0$.

(e) (2 points) How would the obtained T_c value compare with the exact value?

Solution

(a) In the mean-field Hamiltonian, we need to replace the spin interaction term S_iS_j by

$$S_i = m + \delta S_i;$$
$$\delta S_i = S_i - \langle S_i \rangle = S_i - m$$
$$S_iS_j = ((m + \delta S_i)(m + \delta S_j)) = m^2 + m(\delta S_i + \delta S_j) + \cancelto{0}{\delta S_i \delta S_j}$$

Now:

$$\begin{aligned} S_iS_j &= m^2 + m(\delta S_i + \delta S_j) \\ &= m^2 + m(S_i - m) + m(S_j - m) = \\ &= \cancel{m^2} + mS_i - \cancel{m^2} + mS_j - m^2 = \\ &= m((S_i + S_j) - m) \end{aligned}$$

Substituting into the original Hamiltonian:

$$\mathcal{H}_{MF} = -\sum_{\langle i,j \rangle} J_{ij} m((S_i + S_j) - m) - H \sum_i S_i$$

Due to symmetry (all atoms have the same number of nearest neighbors) we have

$$\sum_{\langle i,j \rangle} S_i = \sum_{\langle i,j \rangle} S_j$$
$$\sum_{\langle i,j \rangle} (S_i + S_j) = \sum_{\langle i,j \rangle} 2S_i$$

$$\mathcal{H}_{MF} = -\sum_{\langle i,j \rangle} J_{ij} m(2S_i - m) - H \sum_i S_i$$

The mean-field sum can now be simplified as:

$$\sum_{\langle i,j \rangle} J_{ij} S_i = \frac{1}{2} \sum_{i=1}^{N} \sum_{j \in nn(i)} J_{ij} S_i = \frac{1}{2}(z_1 J_1 + z_2 J_2) \sum_{i=1}^{N} S_i$$

where z_1=6, the number of 1st nearest neighbors and z_2=12. We can name a new mean-field variable $J_0 = (z_1 J_1 + z_2 J_2)$=360K k_B:

$$\sum_{\langle i,j \rangle} J_{ij} S_i = \frac{(z_1 J_1 + z_2 J_2)}{2} \sum_{i=1}^{N} S_i = \frac{1}{2} J_0 \sum_{i=1}^{N} S_i$$

Finally, we have:

$$
\begin{aligned}
\mathcal{H}_{MF} &= -\frac{J_0 m}{2}\sum_i^N (2S_i - m) - H\sum_i S_i = \\
&= \frac{NJ_0 m^2}{2} - J_0 m \sum_i^N (S_i) - H\sum_i S_i = \\
&= \boxed{\frac{NJ_0 m^2}{2} - (J_0 m + H)\sum_i^N S_i}
\end{aligned}
$$

(b) First we need to compute the partition function:

$$
\begin{aligned}
Z_{MF} &= Tr\left(\mathrm{e}^{-\beta\mathcal{H}_{MF}}\right) = \\
&= \prod_i^N \Bigg(\sum_{S_i=\pm2,\pm1,0} e^{-\beta\mathcal{H}_{MF}}\Bigg) = \\
&= \prod_i^N \Bigg(\sum_{S_i=\pm2,\pm1,0}\Bigg)\mathrm{e}^{-\frac{\beta NJ_0 m^2}{2}}\mathrm{e}^{\beta(J_0 m + H)\sum_i S_i} = \\
&= \mathrm{e}^{-\frac{\beta NJ_0 m^2}{2}}\prod_i^N\Bigg(\sum_{S_i=\pm2,\pm1,0}\mathrm{e}^{\beta(J_0 m + H)\sum_i S_i}\Bigg) = \\
&= \mathrm{e}^{-\frac{\beta NJ_0 m^2}{2}}\prod_i^N\Bigg(1 + \mathrm{e}^{\beta(J_0 m+H)} + \mathrm{e}^{-\beta(J_0 m+H)} + \mathrm{e}^{2\beta(J_0 m+H)} + \mathrm{e}^{-2\beta(J_0 m+H)}\Bigg) = \\
&= \mathrm{e}^{-\frac{\beta NJ_0 m^2}{2}}\prod_i^N\Bigg(1 + 2\cosh(\beta(J_0 m + H)) + 2\cosh(2\beta(J_0 m + H))\Bigg) = \\
&= \mathrm{e}^{-\frac{\beta NJ_0 m^2}{2}}\mathcal{G}(m,H)^N
\end{aligned}
$$

Where $\mathcal{G}(m,H) = 1+2\cosh(\beta(J_0 m+H))+2\cosh(2\beta(J_0 m+H))$. From the partition function, the free energy is:

$$
\boxed{\mathcal{F} = -k_\beta T ln\mathcal{Z} = \frac{\beta NJ_0 m^2}{2} - Nk_\beta T\ln\mathcal{G}(m,H)}
$$

(c) Find the mean field equation for m. The mean field eq. for m is obtained at the minimum of $\mathcal{F} \rightarrow \frac{\partial\mathcal{F}}{\partial m} = 0$

$$
\frac{\partial\mathcal{F}}{\partial m} = NJ_0 m - \frac{Nk_\beta T J_0}{k_\beta T}\frac{2\sinh\left(\frac{H+J_0 m}{k_\beta T}\right) + 4\sinh\left(\frac{2H+2J_0 m}{k_\beta T}\right)}{1 + 2\cosh\left(\frac{H+J_0 m}{k_\beta T}\right) + 2\cosh\left(\frac{2H+2J_0 m}{k_\beta T}\right)} = 0
$$

$$
\rightarrow \boxed{m = \frac{2\sinh\left(\frac{H+J_0 m}{k_\beta T}\right) + 4\sinh\left(\frac{2H+2J_0 m}{k_\beta T}\right)}{1 + 2\cosh\left(\frac{H+J_0 m}{k_\beta T}\right) + 2\cosh\left(\frac{2H+2J_0 m}{k_\beta T}\right)}}
$$

(d) We can make the following change of variables (just to simplify the notation): $\theta = \frac{k_\beta T}{J_0}$ and $h = H/J_0$. The expression for m is then:

$$\boxed{m = \frac{2\mathrm{sinh}\left(\frac{m+h}{\theta}\right) + 4\mathrm{sinh}\left(\frac{2m+2h}{\theta}\right)}{1 + 2\mathrm{cosh}\left(\frac{m+H}{\theta}\right) + 2\mathrm{cosh}\left(\frac{2m+2H}{\theta}\right)}}$$

Now, setting the slopes of the LHS and RHS of the equation to be the same at m=0 and H=0 yields the T$_c$ equation. For the LHS the slope is equal to 1. For the RHS we have:

$$\begin{aligned}
\lim_{H,m\to 0} RHS &= \frac{2\left(\frac{m+h}{\theta}\right) + 2\left(\frac{m+h}{\theta}\right)}{5} \\
&= \frac{2}{\theta}(m+h) \\
\frac{\partial \mathrm{RHS}}{\partial m} &= \frac{2}{\theta}
\end{aligned}$$

This means that $\frac{2}{\theta_c} = 1 \Rightarrow \theta_c = 2$, $\boxed{\mathrm{T}_c = 2J_0 = 720K}$

(e) How would the obtained T$_c$ value compare with the exact value? Since the mean field model assumes that fluctuations are small, it generally overestimates the system's tendency to order and thus overestimates the value of T$_c$

Statistical Mechanics 3

Pair production in massive stars

Hydrogen gas in the core of a massive star can be hot enough to produce electron-positron pairs by (for example) electron scattering off of protons,

$$e^- + p \to e^- + p + e^- + e^+.$$

(a) (2 points) If this reaction, and its inverse, are in statistical equilibrium, write down the relation between the chemical potential of the electron, μ_-, and the chemical potential of the positron, μ_+.

(b) (3 points) Suppose that the gas is *relatively* cold, such that the temperature T satisfies,

$$kT \ll m_e c^2,$$

where m_e is the rest mass of the electron, k is Boltzmann's constant, and c is the speed of light. Under these conditions, what is the distribution over momentum p for given chemical potentials $\mu_\pm$, and temperature T?

(c) (3 points) Making the further assumption that the gas is very dilute, find an expression for the physical density of electrons n_- and positrons n_+.

(d) (7 points) The gas is neutral, with proton number density n. Determine the ratio n_+/n in terms of the parameters,

$$y \equiv \frac{1}{4} n\lambda^3 \exp\left(\frac{m_e c^2}{kT}\right), \text{ and } \lambda \equiv \frac{h}{\sqrt{2\pi m_e kT}}.$$

(e) (5 points) Pairs can also be produced in the core of a star from photons $\gamma + \gamma \leftrightarrow e^- + e^+$. If this reaction is dominant, compare qualitatively the $S =$ constant compressibility of a gas of photons containing $e^\pm$ to that of photons only.

Solution

Hydrogen gas in the core of a massive star can be hot enough to produce electron-positron pairs by (for example) electron scattering off of protons,

$$e^- + p \to e^- + p + e^- + e^+.$$

(a) If this reaction, and its inverse, are in statistical equilibrium, write down the relation between the chemical potential of the electron, μ_-, and the chemical potential of the positron, μ_+.

In statistical equilibrium, where the reaction and its inverse have the same rate, $\mu_- + \mu_+ = 0$.

(b) Suppose that the gas is *relatively* cold, such that the temperature T satisfies,

$$kT \ll m_e c^2,$$

where m_e is the rest mass of the electron, k is Boltzmann's constant, and c is the speed of light. Under these conditions, what is the distribution over momentum p for given chemical potentials $\mu_\pm$, and temperature T?

The kinetic energy,

$$E = \mathcal{E} - m_e c^2 \sim kT \ll m_e c^2,$$

with,

$$\mathcal{E} = \sqrt{p^2c^2 + m_e^2 c^4}.$$

In the non-relativistic limit we have,

$$E = \frac{p^2}{2m_e},$$

so,

$$\mathcal{E} = m_e c^2 + \frac{p^2}{2m_e}.$$

The required distribution is then, e.g.,

$$n_-(p) = \frac{2}{h^3}\left[\exp\left(\frac{\mathcal{E} - \mu_-}{kT}\right) + 1\right]^{-1},$$

with h being Planck's constant.

(c) Making the further assumption that the gas is very dilute, find an expression for the physical density of electrons n_- and positrons n_+.

Physically, we just need to integrate over momentum,

$$n = \int n(p) 4\pi p^2 \mathrm{d}p,$$

for either species. Being in the dilute limit means that the exponential terms in $n_\pm(p)$ dominate, and we drop the "1". The required integral is,

$$n_- = \frac{2}{h^3}\exp\left[\frac{\mu_- - m_e c^2}{kT}\right]\int 4\pi p^2 \exp\left[\frac{-p^2}{2m_e kT}\right] \mathrm{d}p.$$

We need to evaluate a standard integral of the form (for some constant c),

$$\int p^2 \exp\left(-p^2/c\right) \mathrm{d}p = (1/4)\sqrt{\pi}c^{3/2}.$$

With this, the physical density evaluates to,

$$n_\pm = \frac{2}{h^3}\left(2\pi m_e kT\right)^{3/2}\exp\left(\frac{\mu_\pm - m_e c^2}{kT}\right).$$

(d) The gas is neutral, with proton number density n. Determine the ratio n_+/n in terms of the parameters,

$$y \equiv \frac{1}{4}n\lambda^3 \exp\left(\frac{m_e c^2}{kT}\right), \text{ and } \lambda \equiv \frac{h}{\sqrt{2\pi m_e kT}}.$$

Charge neutrality implies,

$$n + n_+ = n_-,$$

and we have from part (i),

$$\mu_- = -\mu + .$$

We simplify the notation by writing the result from part (iii) in terms of the suggested variables, so that,

$$n_+ = \frac{n}{2y}e^{\mu_+/kT},$$

and,

$$n_- = \frac{n}{2y}e^{\mu_-/kT} = \frac{n}{2y}e^{-\mu_+/kT}.$$

Using charge neutrality, the quantity we want is,

$$\frac{n_+}{n} = \frac{n_-}{n} - 1.$$

Substituting,

$$\frac{n_+}{n} = \frac{n}{4n_+ y^2} - 1.$$

In terms of $t \equiv n_+/n$ we have a quadratic,

$$t^2 + t - \frac{1}{4y^2} = 0.$$

The solution in terms of the suggested variables is,

$$\frac{n_+}{n} = \frac{1}{2y\left(y + \sqrt{1+y^2}\right)}.$$

Equivalent solutions are fine too of course.

(e) Pairs can also be produced in the core of a star from photons $\gamma + \gamma \leftrightarrow e^- + e^+$. If this reaction is dominant, compare qualitatively the $S =$ constant compressibility of a gas of photons containing $e^\pm$ to that of photons only.

The onset of pair production softens (makes more compressible) the effective equation of state. There are various ways to argue this qualitatively. e.g. we can note that for a system where radiation pressure is dominant, but there are no pairs, $\Gamma \simeq 4/3$. *One pairs are being produced at a high rate, energy goes into the rest mass of the pairs and does not contribute as much pressure as if it remained in the photons. So* $\Gamma < 4/3$.

This physics is thought to lead to the hypothesized class of pair-instability supernovae, and to affect the distribution of masses of stellar mass black holes that can be observed (in mergers) via gravitational waves.